Product Theorem for Quaternion Fourier Transform

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Abstract

This paper presents in some detail the quaternion Fourier transform (QFT) of the product of two quaternion functions. It is shown that the proposed product theorem for the QFT is closely related to the convolution in the quaternion Fourier domain.

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1 Introduction

The quaternion Fourier transform (QFT) which is considered as a generalization of the classical Fourier transform (FT) has recently been extensively used and discussed as a very efficient mathematical tool in signal processing for signals [1, 5]. Many properties of generalized transform are already known, such as translation, modulation, differentiation, convolution, correlation, the Parseval and Plancherel formula, and uncertainty principle (see, for example, [2, 3, 4]). The properties are extensions of the corresponding version of the FT with the same modifications. The most important property of the QFT for signal processing applications is convolution theorem. It describes the relationship between convolution of two quaternion functions and the QFTs. This property is in fact closely related to the product theorem in the quaternion Fourier domain.
It is well known that in the Fourier domain the product theorem states that the Fourier transform of the product of two real and complex functions is the convolution of their Fourier transforms. In this paper we propose an extension of the product theorem for the QFT. This property describes how the QFT relates to product of two quaternion functions.

2 Quaternion

The quaternion, which is a type of hypercomplex number, was formally introduced by Hamilton in 1843. It is a generalization of complex number to a 4D algebra and is denoted by $\mathbb{H}$. Every element of $\mathbb{H}$ can be written in a hypercomplex form as follows

$$\mathbb{H} = \{ q = q_0 + iq_1 + jq_2 + kq_3 : q_0, q_1, q_2, q_3 \in \mathbb{R} \}. \quad (1)$$

Here the three different imaginary parts obey the following multiplication rules:

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1. \quad (2)$$

For a quaternion $q = q_0 + iq_1 + jq_2 + kq_3 \in \mathbb{H}$, $q_0$ is called the scalar part of $q$ denoted by $\text{Sc}(q)$ and a pure quaternion $q$ denoted by $\text{Vec}(q) = iq_1 + jq_2 + kq_3$.

Any quaternion $q$ can be written as

$$q = |q| e^{\mu \theta}, \quad |q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}, \quad (3)$$

where $\theta = \arctan |\text{Sc}(q)|/\text{Vec}(q)$, $0 \leq \theta \leq \pi$ is the eigen angle or phase of $q$ and $\mu$ is any pure unit quaternion such that $\mu^2 = -1$ When $|q| = 1$, $q$ is a unit quaternion.

**Proposition 2.1.** If $p$ and $q$ are two pure quaternions, then

- $p$ and $q$ are parallel ($p \parallel q$) if and only if $pq = qp$,
- $p$ and $q$ are perpendicular ($p \perp q$) if and only if $pq = -qp$.

3 Main Results

In this section, we begin by introducing a definition of the quaternion Fourier transform (QFT).

**Definition 3.1 (QFT).** Let $f$ be in $L^2(\mathbb{R}^2; \mathbb{H})$. Then quaternion Fourier transform of the function $f$ is given by

$$\mathcal{F}_q \{ f \} (\omega) = \int_{\mathbb{R}^2} f(x) e^{-i\omega \cdot x} dx, \quad dx = dx_1 dx_2, \quad (4)$$

where $\omega, x \in \mathbb{R}^2$. 
Theorem 3.2 (Inverse QFT). Suppose that $f$ be in $L^2(\mathbb{R}^2; \mathbb{H})$ and $\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2; \mathbb{H})$. Then inverse transform of the QFT is given by

$$f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\omega) e^{\mu x \cdot \omega} d\omega.$$  \hfill (5)

Definition 3.3 (Quaternion Convolution). Let $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$. The convolution of two quaternion functions $f$ and $g$ is denoted by $f \ast g$ and is defined by

$$(f \ast g)(x) = \int_{\mathbb{R}^2} f(t)g(x - t) \, dt.$$ \hfill (6)

It is not difficult to see that

$$(f \ast g)(-x) = \int_{\mathbb{R}^2} f(t)g(t - x) \, dt.$$ \hfill (7)

Definition 3.4 (Quaternion Correlation). Let $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ be two quaternion functions. The correlation of $f$ and $g$ is defined by

$$(f \circ g)(x) = \int_{\mathbb{R}^2} \overline{f(y)}g(x + y) \, dy.$$ \hfill (8)

The main result of this paper is the following theorem, which describes the relationship between the product of two quaternion functions and its QFT.

Theorem 3.5. Let $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$. Then the QFT of product of two quaternion functions $f$ and $g$ is given by

$$\mathcal{F}_q\{fg\}(\omega) = \frac{1}{(2\pi)^2} \left( (\mathcal{F}_q\{g\} * \mathcal{F}_q\{f_0\})(\omega) + i(\mathcal{F}_q\{g\} * \mathcal{F}_q\{f_1\})(\omega) 
+ j(\mathcal{F}_q\{g\} * \mathcal{F}_q\{f_2\})(\omega) + k(\mathcal{F}_q\{g\} * \mathcal{F}_q\{f_3\})(\omega) \right).$$ \hfill (9)

Proof. Applying the QFT definition (4) yields

$$\mathcal{F}_q\{fg\}(\omega)$$

$$= \int_{\mathbb{R}^2} f(x)g(x) e^{-\mu x \cdot \omega} \, dx$$

$$= \int_{\mathbb{R}^2} f(x) \left( \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q\{g\}(u) e^{i\mu u \cdot x} \, du \right) e^{-\mu x \cdot \omega} \, dx$$

$$= \int_{\mathbb{R}^2} (f_0(x) + if_1(x) + jf_2(x) + kf_3(x))$$

$$\times \left( \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q\{g\}(u) e^{i\mu u \cdot x} \, du \right) e^{-\mu x \cdot \omega} \, dx$$

$$= \frac{1}{(2\pi)^2} \left( \int_{\mathbb{R}^2} f_0(x) \mathcal{F}_q\{g\}(u) e^{i\mu u \cdot x} \, du \right) e^{-\mu x \cdot \omega} \, dx$$

$$+ \frac{i}{(2\pi)^2} \left( \int_{\mathbb{R}^2} f_1(x) \mathcal{F}_q\{g\}(u) e^{i\mu u \cdot x} \, du \right) e^{-\mu x \cdot \omega} \, dx$$

$$+ \frac{j}{(2\pi)^2} \left( \int_{\mathbb{R}^2} f_2(x) \mathcal{F}_q\{g\}(u) e^{i\mu u \cdot x} \, du \right) e^{-\mu x \cdot \omega} \, dx$$

$$+ \frac{k}{(2\pi)^2} \left( \int_{\mathbb{R}^2} f_3(x) \mathcal{F}_q\{g\}(u) e^{i\mu u \cdot x} \, du \right) e^{-\mu x \cdot \omega} \, dx.$$

Thus, we have

$$\mathcal{F}_q\{fg\}(\omega) = \frac{1}{(2\pi)^2} \left( (\mathcal{F}_q\{g\} * \mathcal{F}_q\{f_0\})(\omega) + i(\mathcal{F}_q\{g\} * \mathcal{F}_q\{f_1\})(\omega) 
+ j(\mathcal{F}_q\{g\} * \mathcal{F}_q\{f_2\})(\omega) + k(\mathcal{F}_q\{g\} * \mathcal{F}_q\{f_3\})(\omega) \right).$$
\[ \begin{align*}
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q(g)(u) \mathcal{F}_q(f_0)(v) e^{iuv} e^{j\omega u - j\omega v}\, dv \, du \, dx \\
&+ i \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q(g)(u) \mathcal{F}_q(f_1)(v) e^{iuv} e^{j\omega u - j\omega v}\, dv \, du \, dx \\
&+ j \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q(g)(u) \mathcal{F}_q(f_2)(v) e^{iuv} e^{j\omega u - j\omega v}\, dv \, du \, dx \\
&+ k \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q(g)(u) \mathcal{F}_q(f_3)(v) e^{iuv} e^{j\omega u - j\omega v}\, dv \, du \, dx \\
&= \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q(g)(u) \mathcal{F}_q(f_0)(v) \delta(v + u - \omega)\, dv \, du \\
&+ i \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q(g)(u) \mathcal{F}_q(f_1)(v) \delta(v + u - \omega)\, dv \, du \\
&+ j \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q(g)(u) \mathcal{F}_q(f_2)(v) \delta(v + u - \omega)\, dv \, du \\
&+ k \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q(g)(u) \mathcal{F}_q(f_3)(v) \delta(v + u - \omega)\, dv \, du \\
&= \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q(g)(u) \mathcal{F}_q(f_0)(\omega - u)\, du \\
&+ i \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q(g)(u) \mathcal{F}_q(f_1)(\omega - u)\, du \\
&+ j \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q(g)(u) \mathcal{F}_q(f_2)(\omega - u)\, du \\
&+ k \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q(g)(u) \mathcal{F}_q(f_3)(\omega - u)\, du,
\end{align*} \]

which completes the proof of the theorem. \( \square \)

As an immediate consequence of the above theorem, we get the following corollary.

**Corollary 3.6.** Let \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \). Assume that the QFT of \( g \) is a real-
valued function, then Theorem 3.5 will reduce to
\[ F_q\{fg\}(\omega) = \frac{1}{(2\pi)^2}(F_q\{f\} \ast F_q\{g\})(\omega). \] (10)

Proof. An application of the QFT definition (4) we easily obtain
\[
F_q\{fg\}(\omega) = \int_{\mathbb{R}^2} f(x)g(x)e^{-i\omega \cdot x} \, dx \\
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} F_q\{f\}(v)F_q\{g\}(u)e^{i\mu \cdot x} e^{i\omega \cdot x} du \, dv \, dx
\]
The assumption allows us to interchange the position of kernel \( e^{i\mu \cdot x} \) and the function \( F_q\{g\} \). Therefore, we get
\[
F_q\{fg\}(\omega) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} F_q\{f\}(v)F_q\{g\}(u)\delta(v + u - \omega) du \, dv \\
= \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} F_q\{f\}(v)F_q\{g\}(\omega - v) dv.
\]
As desired. \( \square \)

The following theorem provides an alternative form of Theorem 3.5.

**Theorem 3.7.** Let \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \). If \( g = g \) is a pure quaternion function, then we have
\[
F_q\{fg\}(\omega) = \frac{1}{(2\pi)^2} \left( (F_q\{f\} \ast F_q\{g_{\parallel,\mu}\})(\omega) + (F_q\{f\} \ast F_q\{g_{\perp,\mu}\})(-\omega) \right).
\]

Proof. Because \( g \) is a pure quaternion function, then using Preposition 2.1 we may decompose \( g \) with respect to the axis \( \mu \) into \( g_{\parallel,\mu} + g_{\perp,\mu} \). It means that we have
\[
F_q\{fg\}(\omega) = \int_{\mathbb{R}^2} f(x)g(x)e^{-i\omega \cdot x} \, dx \\
= \int_{\mathbb{R}^2} \left( \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} F_q\{f\}(v)e^{i\mu \cdot x} \, du \right) g(x)e^{-i\omega \cdot x} \, dx
\]
\[
\begin{align*}
&= \int_{\mathbb{R}^2} \left( \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q \{ f \} (v) e^{i\omega \cdot x} \, du \right) \left( g_{\parallel,\mu}(x) + g_{\perp,\mu}(x) \right) e^{-i\omega \cdot x} \, dx \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \frac{1}{(2\pi)^2} \mathcal{F}_q \{ g \} (u) g_{\parallel,\mu}(x) \right) e^{i\omega \cdot x} e^{-i\omega \cdot x} \, dv \, dx \\
&\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \frac{1}{(2\pi)^2} \mathcal{F}_q \{ g \} (u) g_{\perp,\mu}(x) \right) e^{i\omega \cdot x} e^{-i\omega \cdot x} \, dv \, dx \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \frac{1}{(2\pi)^4} \mathcal{F}_q \{ g \} (u) \mathcal{F}_q \{ g_{\parallel,\mu} \} (v) \right) e^{i\omega \cdot x} e^{i\omega \cdot x} e^{-i\omega \cdot x} \, du \, dv \, dx \\
&\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \frac{1}{(2\pi)^4} \mathcal{F}_q \{ g \} (u) \mathcal{F}_q \{ g_{\perp,\mu} \} (v) \right) e^{i\omega \cdot x} e^{-i\omega \cdot x} \, du \, dv \, dx \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \frac{1}{(2\pi)^4} \mathcal{F}_q \{ g \} (u) \mathcal{F}_q \{ g_{\parallel,\mu} \} (v) \right) \delta(v + u - \omega) \, dv \, du \\
&\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \frac{1}{(2\pi)^4} \mathcal{F}_q \{ g \} (u) \mathcal{F}_q \{ g_{\perp,\mu} \} (v) \right) \delta(u - v - \omega) \, dv \, du \\
&= \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q \{ g \} (u) \mathcal{F}_q \{ g_{\parallel,\mu} \} (\omega - u) \, dv \\
&\quad + \int_{\mathbb{R}^2} \frac{1}{(2\pi)^2} \mathcal{F}_q \{ g \} (u) \mathcal{F}_q \{ g_{\perp,\mu} \} (u - \omega) \, dv.
\end{align*}
\]

This proves the theorem. \(\square\)

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**References**


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