On Two-Dimensional Quaternion Wigner-Ville Distribution

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Abstract

In this paper, we present the two-dimensional quaternion Wigner-Ville distribution (QWVD). The transform is constructed by substituting the Fourier transform kernel with the quaternion Fourier transform (QFT) kernel in the classical Wigner-Ville distribution definition. Based on the properties of quaternions and the QFT kernel we obtain three types of the QWVD. We discuss some useful properties of various definitions for the QWVD, which are extensions of the classical Wigner-Ville distribution properties.

Keywords: quaternion-valued function, quaternion Fourier transform, quaternion Wigner-Ville distribution

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1 Introduction

The classical Wigner-Ville distribution (WVD) or Wigner-Ville transform (WVT) is an important tool in the time-frequency signal analysis. It was first introduced by Eugene Wigner in his calculation of the quantum corrections of classical statistical mechanics. It was independently derived again by J. Ville in 1948 as a quadratic representation of the local time-frequency energy of a signal. In [5, 6, 10], the authors introduced the WVT and established some important properties of the WVT. The transform is then extended to the linear canonical transform (LCT) domain by replacing the kernel of the classical Fourier transform (FT) with the kernel of the LCT in the WVD domain [3].

As a generalization of the real and complex Fourier transform (FT), the quaternion Fourier transform (QFT) has been of interest to researchers for some years. A number of useful properties of the QFT have been found including shift, modulation, convolution, correlation, differentiation, energy conservation, uncertainty principle and so on. Due to the non-commutative property of quaternion multiplication, there are three different types of two-dimensional QFTs. These three QFTs are so-called a left-sided QFT, a right-sided QFT, and a two-sided QFT, respectively, (see, for example, [1, 2, 11, 16, 21]). In [7, 8], special properties of the asymptotic behaviour of the right-sided QFT are discussed and generalization of the classical Bohner-Millos theorems to the framework of quaternion analysis are established. Many generalized transforms are closely related to the QFTs. For example the quaternion wavelet transform, fractional quaternion Fourier transform, quaternion linear canonical transform, and quaternionic windowed Fourier transform [4, 12, 13, 14, 15, 17, 18]. Based on the QFTs, one also
may extend the WVD to the quaternion algebra while enjoying similar properties as in the
classical case.

Therefore, the main purpose of this paper is to propose a generalization of the classical
WVD to quaternion algebra, which we call the quaternion Wigner-Ville distribution (QWVD).
Our generalization is constructed by substituting the kernel of the FT with the kernel of the
QFT in the classical WVT definition. Due to the non-commutative rule of quaternions and
the QFT kernel we obtain the definition of different types of the QWVD. We then derive
some important properties of the QWVD such as shift, reconstruction formula, modulation,
and orthogonality relation in detail. We present an example to show the difference between
the QWVD and the WVD.

The organization of the paper is as follows. The remainder of this section introduces some
notations and briefly recalls some general definitions, basic properties of quaternion algebra
and quaternion Fourier transform. In section 3, we provide the basic ideas for the construction
of the two-sided QWVD and derive several its important properties using the two-sided QFT.
The construction of the right-sided QWVD is provided in Section 4. In section 5 we introduce
quaternion ambiguity function (QAF); its important properties are also discussed in this
section.

2 Basics

2.1 Quaternion algebra

The quaternion algebra was formally introduced by Hamilton in 1843, and it is a generalization
of complex numbers. The quaternion algebra over $\mathbb{R}$, denoted by $\mathbb{H}$, is an associative non-
commutative four-dimensional algebra,

$$\mathbb{H} = \{ q = q_0 + iq_1 + jq_2 + kq_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R} \},$$

which obey Hamilton’s multiplication rules

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1.$$ \hspace{1cm} (2)

The quaternion conjugate of a quaternion $q$ is given by

$$\bar{q} = q_0 - iq_1 - jq_2 - kq_3, \quad q_0, q_1, q_2, q_3 \in \mathbb{R},$$ \hspace{1cm} (3)

and it is an anti-involution, i.e.

$$\bar{qp} = \bar{p}\bar{q}.$$ \hspace{1cm} (4)

From (3) we obtain the norm of $q \in \mathbb{H}$ defined as

$$|q| = \sqrt{qq} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$ \hspace{1cm} (5)

It is not difficult to see that

$$|qp| = |q||p|, \quad \forall p, q \in \mathbb{H}.$$ \hspace{1cm} (6)

Using the conjugate (3) and the modulus of $q$, we can define the inverse of $q \in \mathbb{H} \setminus \{0\}$ as

$$q^{-1} = \frac{\bar{q}}{|q|^2}$$ \hspace{1cm} (7)

which shows that $\mathbb{H}$ is a normed division algebra.
We use parenthesis (,) to denote the inner product of two quaternion functions, \( f, g : \mathbb{R}^2 \rightarrow \mathbb{H} \), as follows:
\[
(f, g)_{L^2(\mathbb{R}^2; \mathbb{H})} = \int_{\mathbb{R}^2} f(x)\overline{g(x)} \, d^2x.
\]
When \( f = g \) we obtain the associated norm
\[
\|f\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2 = \int_{\mathbb{R}^2} |f(x)|^2 \, d^2x.
\]
As a consequence of the inner product (8) we obtain the quaternion Cauchy-Schwarz inequality
\[
\left| \int_{\mathbb{R}^2} fg \, d^2x \right| \leq \left( \int_{\mathbb{R}^2} |f|^2 d^2x \right)^{1/2} \left( \int_{\mathbb{R}^2} |g|^2 d^2x \right)^{1/2}, \quad \forall f, g \in L^2(\mathbb{R}^2; \mathbb{H}).
\]

2.2 Quaternion Fourier Transform (QFT)

The quaternion (or hypercomplex) Fourier transform is defined similar to the classical FT of the 2D funtions. The noncommutative property of quaternion multiplication allows us to have three different definitions of the QFT. In the following we briefly introduce the two-sided QFT and the right-sided QFT. For more details we refer the reader to \([11, 16, 22]\).

**Definition 2.1 (Two-Sided QFT).** The two-sided QFT of \( f \in L^2(\mathbb{R}^2; \mathbb{H}) \) is the transform \( \mathcal{F}_q^D \{ f \} \in L^2(\mathbb{R}^2, \mathbb{H}) \) given by the integral
\[
\mathcal{F}_q^D \{ f \}(\omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 x_1} f(x)e^{-i\omega_2 x_2} d^2x,
\]
where \( x = x_1e_1 + x_2e_2, \omega = \omega_1e_1 + \omega_2e_2 \), and the quaternion exponential product \( e^{-i\omega_1 x_1}e^{-i\omega_2 x_2} \) is called the quaternion Fourier kernel.

**Theorem 2.1 (Inverse two-sided QFT).** Suppose that \( f \in L^2(\mathbb{R}^2; \mathbb{H}) \) and \( \mathcal{F}_q^D \{ f \} \in L^1(\mathbb{R}^2; \mathbb{H}) \). Then the two-sided QFT of \( f \) is an invertible transform and its inverse is given by
\[
f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\omega_1 x_1} \mathcal{F}_q^D \{ f \}(\omega)e^{i\omega_2 x_2} d^2\omega,
\]
where the quaternion exponential product \( e^{i\omega_2 x_2}e^{i\omega_1 x_1} \) is called the inverse two-sided quaternion Fourier kernel.

Notice that if \( \mathcal{F}_q^D \{ f \}(\omega) = 1 \), from the inverse of the two-sided QFT we obtain the quaternion Dirac’s delta function, i.e.
\[
\delta(x) = \delta(x_1)\delta(x_2) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\omega_1 x_1} e^{i\omega_2 x_2} d^2\omega.
\]

We have known that Parseval’s formula is not valid for the two-sided QFT, but a special case of the Parseval’s formula called the Plancherel’s formula remains valid, i.e.,
\[
\|f\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2 = \frac{1}{(2\pi)^2} \|\mathcal{F}_q^D \{ f \}\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2.
\]

We introduce Parseval’s formula for the right sided QFT as
\[
(f, g)_{L^2(\mathbb{R}^2; \mathbb{H})} = \frac{1}{(2\pi)^2} \langle \mathcal{F}_q^D \{ f \}, \mathcal{F}_q^R \{ g \} \rangle_{L^2(\mathbb{R}^2; \mathbb{H})},
\]
where the right-sided QFT is defined by for every \( f \in L^2(\mathbb{R}^2; \mathbb{H}) \)
\[
\mathcal{F}_q^R \{ f \}(\omega) = \int_{\mathbb{R}^2} f(x)e^{-i\omega_1 x_1}e^{-i\omega_2 x_2} d^2x.
\]
2.3 Fundamental Operators

Before we discuss the QWVD we need to introduce some notations, which will be used in the next section. For a quaternion function \( f \in L^2(\mathbb{R}^2; \mathbb{H}) \) we define the translation, modulation and dilation as follows:

\[
T_a f(x) = f(x - a), \quad M_{\omega_0} f(x) = e^{i\omega_0 x_1} f(x) e^{j\omega_0 x_2}, \quad D_c f(x) = \frac{1}{c} f\left(\frac{x}{c}\right),
\]

(17)

where \( \omega_0 = u_0 e_1 + v_0 e_2 \) and \( c \in \mathbb{R}^+ \). The composition of the translation and modulation is called the time-frequency shift, i.e.,

\[
M_{\omega_0} T_a f(x) = e^{i\omega_0 x_1} f(x - a) e^{j\omega_0 x_2}, \quad a \in \mathbb{R}^2.
\]

(18)

Just as the classical case (see [10]), we obtain the canonical commutation relations

\[
T_a M_{\omega_0} f = e^{-i\omega_0 a_1} M_{\omega_0} T_a f e^{-j\omega_0 a_2}.
\]

(19)

3 Quaternion Wigner-Ville Distribution (QWVD)

In this section, we introduce the 2-D Quaternion Wigner-Ville distribution (QWVD). We investigate several basic properties of the QWVD which are important for signal representation in signal processing.

3.1 Definition of 2-D QWVD

Based on the properties of quaternions and the definition of the classical Wigner-Ville transform associated with the Fourier transform, we obtain a definition of the QWVD by replacing the kernel of the FT with the kernel of the two-sided QFT in the classical WVD definition as follows.

**Definition 3.1.** The cross two-sided quaternion Wigner-Ville distribution of two-dimensional functions (or signals) \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \) is given by

\[
W^D_{f,g}(t, \omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 \tau_1} f(t + \frac{\tau_1}{2}) \bar{g}(t - \frac{\tau_1}{2}) e^{-j\omega_2 \tau_2} d^2 \tau,
\]

(20)

provided the integral exists.

It should be remembered that the kernel of the cross two-sided QWVD in (20) does not commute with quaternion functions \( f \) and \( g \) so that several properties of the WVD are not valid in the cross two-sided QWVD.

By making the change of variables \( t + \frac{\tau_1}{2} = x \), equation (20) can be written in the form

\[
W^D_{f,g}(t, \omega) = 4e^{i2\omega_1 t_1} \int_{\mathbb{R}^2} e^{-i\omega_1(2x_1-2t_1)} f(x) \bar{g}(2t - x) e^{-j\omega_2(4x_2-2t_2)} d^2 x
\]

\[
= 4e^{i2\omega_1 t_1} \int_{\mathbb{R}^2} e^{-i2\omega_1 x_1} f(x) \bar{g}(2t - x) e^{-j2\omega_2 x_2} d^2 x e^{j2\omega_2 t_2}
\]

\[
= 4e^{i2\omega_1 t_1} \int_{\mathbb{R}^2} e^{-i2\omega_1 x_1} f(x) \bar{g}(-(x - 2t)) e^{-j2\omega_2 x_2} d^2 x e^{j2\omega_2 t_2}.
\]

(21)
We see that the above expression gives an equivalent definition of $\mathcal{W}^D_{f,g}(t,\omega)$ in the form

$$\mathcal{W}^D_{f,g}(t,\omega) = 4e^{i2\omega_1 t_1}Q_{1g} f(2t, 2\omega) e^{i2\omega_2 t_2}, \quad (22)$$

where $Q_{1g} x = g(-x)$. Here $Q_{1g} f(2t, 2\omega)$ is the windowed quaternionic Fourier transform which was recently proposed by Fu et al. [12]. Furthermore, if we write $R_t(\tau) = f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2})$, we immediately obtain

$$\mathcal{W}^D_{f,g}(t,\omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 \tau_1} R_t(\tau)e^{-i\omega_2 \tau_2} d^2 \tau$$

which tell us that the the cross two-sided QWVD is in fact the two-sided QFT of the function $R_t(\tau) = f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2})$ with respect to $\tau$. This fact is very important in proving the Moyal’s formula for the two-sided QWVD.

**Lemma 3.1.** Let $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ be two quaternion-valued functions. Then $\mathcal{W}^D_{f,g}(t,\omega)$ is bounded on $L^2(\mathbb{R}^2; \mathbb{H})$, i.e.,

$$\left| \mathcal{W}^D_{f,g}(t,\omega) \right| \leq 4 \|f\|_{L^2(\mathbb{R}^2; \mathbb{H})} \|g\|_{L^2(\mathbb{R}^2; \mathbb{H})}. \quad (24)$$

In particular, if $f = g$, the above expression reduces to

$$\left| \mathcal{W}^D_{f}(t,\omega) \right| \leq 4 \mathcal{W}^D_{f}(0,0). \quad (25)$$

**Proof.** Application of the quaternion Cauchy-Schwarz inequality (10) we easily obtain

$$\left| \mathcal{W}^D_{f,g}(t,\omega) \right|^2 = \left| \int_{\mathbb{R}^2} e^{-i\omega_1 \tau_1} f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) e^{-i\omega_2 \tau_2} d^2 \tau \right|^2$$

$$\leq \int_{\mathbb{R}^2} \left| e^{-i\omega_1 \tau_1} f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) e^{-i\omega_2 \tau_2} \right|^2 d^2 \tau$$

$$= \int_{\mathbb{R}^2} \left| f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) \right|^2 d^2 \tau$$

$$\leq \int_{\mathbb{R}^2} \left| f(t + \frac{\tau}{2}) \right|^2 d^2 \tau \int_{\mathbb{R}^2} \left| \bar{g}(t - \frac{\tau}{2}) \right|^2 d^2 \tau$$

$$= 16 \int_{\mathbb{R}^2} \left| f(x) \right|^2 d^2 x \int_{\mathbb{R}^2} \left| \bar{g}(y) \right|^2 d^2 y. \quad (26)$$

It means that for all $t$ and $\omega$ we have

$$\left| \mathcal{W}^D_{f,g}(t,\omega) \right| \leq 4 \|f\|_{L^2(\mathbb{R}^2; \mathbb{H})} \|g\|_{L^2(\mathbb{R}^2; \mathbb{H})}, \quad (27)$$

which was to be proved. \hfill \square

The following theorem shows that the cross two-sided QWVD is invertible, that is, the original quaternion signal $f$ can be uniquely determined in terms of its cross two-sided QWVD within a constant factor.

**Theorem 3.2** (Reconstruction formula for the two-sided QWVD). The inverse transform of the cross two-sided QWVD of the signal $f \in L^2(\mathbb{R}^2; \mathbb{H})$ is given by

$$f(t) = \frac{1}{(2\pi)^2 \bar{g}(0)} \int_{\mathbb{R}^2} e^{i\omega_1 t_1} \mathcal{W}^D_{f,g}(\frac{t}{2},\omega) e^{i\omega_2 t_2} d^2 \omega, \quad (28)$$

provided $\bar{g}(0) \neq 0$. 

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Proof. From the definition of the cross two-sided QWVD, we know that

\[ W_{f,g}^D(t, \omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 \tau_1} f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) e^{-i\omega_2 \tau_2} d^2\tau. \]  

(29)

Indeed, from the inverse of the two-sided QFT (12), it follows that

\[ f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\omega_1 \tau_1} W_{f,g}^D(t, \omega) e^{i\omega_2 \tau_2} d^2\omega. \]

(30)

Letting \( \frac{\tau}{2} = t \), the above expression will lead to

\[ f(2t) \bar{g}(0) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{2i\omega_1 t_1} W_{f,g}^D(t, \omega) e^{2i\omega_2 t_2} d^2\omega, \]

(31)

and the final result can be obtained by letting \( 2t = s \), i.e.,

\[ f(s) = \frac{1}{(2\pi)^2} \bar{g}(0) \int_{\mathbb{R}^2} e^{i\omega_1 s_1} W_{f,g}^D(\frac{s}{2}, \omega) e^{i\omega_2 s_2} d^2\omega, \]

(32)

which completes the proof. \( \square \)

**Lemma 3.3.** For any \( t \) and \( \omega \), if \( \|f\|_{L^2(\mathbb{R}^2; \mathbb{H})} = 1 \) and \( \|g\|_{L^2(\mathbb{R}^2; \mathbb{H})} = 1 \), then we obtain

\[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| W_{f,g}^D(t, \omega) \right|^2 d^2t d^2\omega = (4\pi)^2. \]

(33)

Proof. Since \( W_{f,g}^D(t, \omega) \) is the two-sided QFT of \( R_t(\tau) \), equation (14) yields

\[ \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \left| W_{f,g}^D(t, \omega) \right|^2 d^2\omega = \int_{\mathbb{R}^2} \left| R_t(\tau) \right| d^2\tau = \int_{\mathbb{R}^2} \left| f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) \right|^2 d^2\tau. \]

(34)

Using the change of variables \( x = t + \frac{\tau}{2}, y = t - \frac{\tau}{2} \) and then integrating (34) with respect to \( d^2t \) we immediately

\[ \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| W_{f,g}^D(t, \omega) \right|^2 d^2t d^2\omega = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) \right|^2 d^2t d^2\tau \]

\[ = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| f(t + \frac{\tau}{2}) \right|^2 \left| \bar{g}(t - \frac{\tau}{2}) \right|^2 d^2t d^2\tau \]

\[ = 4 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| f(x) \right|^2 \left| \bar{g}(y) \right|^2 d^2y d^2t. \]

(35)

This finishes the proof. \( \square \)

**Remark 3.1.** Equation (33) is known as the radar uncertainty principle in the cross two-sided QWVD domain. It shows that the quaternion function \( W_{f,g}^D(t, \omega) \) cannot be concentrated arbitrarily close the the origin.

We obtain the following results which correspond to classical WVD properties (compare to [5, 6]).

**Lemma 3.4.** For \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \), the two-sided QFT of (20) with respect to \( \omega \) can be represented in the form

\[ \mathcal{F}_q(W_{f,g}^D)(t, \sigma) = \int_{\mathbb{R}^2} e^{-i\omega_1 \sigma_1} W_{f,g}^D(t, \omega) e^{-i\omega_2 \sigma_2} d^2\omega = (2\pi)^2 f(t - \frac{\sigma}{2}) \bar{g}(t + \frac{\sigma}{2}). \]

(36)
Proof. A simple computation gives for every $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$
\[
\mathcal{F}_q \{ \mathcal{W}^D_{f,g}(t, \sigma) \} = \left\{ \int_{\mathbb{R}^2} e^{-i\nu_1 \sigma_1} \mathcal{W}_{f,g}(t, \omega) e^{-i\nu_2 \sigma_2} d^2 \omega \right\}
\]
\[
= \left\{ \int_{\mathbb{R}^2} e^{-i\nu_1 \tau_1} R_t(\tau) e^{-i\nu_2 \tau_2} d^2 \tau e^{-i\nu_2 \sigma_2} d^2 \omega \right\}
\]
\[
= \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i\nu_1 (\tau_1 - \sigma_1)} R_t(\tau) e^{i\nu_2 \tau_2} d^2 \omega d^2 \tau \right\}
\]
\[
= (2\pi)^2 \int_{\mathbb{R}^2} \delta(-\tau_1 - \sigma_1) R_t(\tau) \delta(-\tau_2 - \sigma_2) d^2 \tau
\]
\[
= (2\pi)^2 R_t(-\sigma) = (2\pi)^2 f(t - \frac{\sigma}{2}) \bar{g}(t + \frac{\sigma}{2}), \tag{37}
\]
where as usual, $\delta$ denotes the Dirac’s delta function (13). \hfill \Box

It is not difficult to see that for $f = g$ we get the auto two-sided quaternion Wigner-Ville distribution defined by
\[
\mathcal{W}^D_{f,f}(t, \omega) = \mathcal{W}^D_f(t, \omega) = \int_{\mathbb{R}^2} e^{-i\nu_1 \tau_1} f(t + \frac{\tau}{2}) f(t - \frac{\tau}{2}) e^{-i\nu_2 \tau_2} d^2 \tau. \tag{38}
\]
Both the cross two-sided quaternion Wigner-Ville distribution and the auto two-sided Wigner-Ville distribution are often so-called as the two-sided Wigner-Ville distribution or two-sided quaternion Wigner-Ville transform (QWVT). If $f$ is a real function, then the change of variables $\tau = -x$ to (38) yields
\[
\mathcal{W}^D_f(t, \omega) = \int_{\mathbb{R}^2} e^{i\nu_1 x_1} f(t - \frac{x}{2}) f(t + \frac{x}{2}) e^{i\nu_2 x_2} d^2 x
\]
\[
= \int_{\mathbb{R}^2} e^{i\nu_1 x_1} f(t + \frac{x}{2}) f(t - \frac{x}{2}) e^{i\nu_2 x_2} d^2 x
\]
\[
= \mathcal{W}^D_f(t, -\omega). \tag{39}
\]

Lemma 3.5 (Time energy density). If $f \in L^2(\mathbb{R}^2; \mathbb{H})$, then the two-sided QWVD satisfies the time energy density as
\[
\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{W}^D_f(t, \omega) d^2 t d^2 \omega = \|f\|^2_{L^2(\mathbb{R}^2; \mathbb{H})}. \tag{40}
\]
Proof. A direct calculation shows that
\[
\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i\nu_1 \sigma_1} \mathcal{W}^D_{f,g}(t, \omega) e^{-i\nu_2 \sigma_2} d^2 \omega = f(t - \frac{\sigma}{2}) \bar{g}(t + \frac{\sigma}{2}). \tag{41}
\]
Changing variables $t + \frac{\sigma}{2} = s$ and $t - \frac{\sigma}{2} = z$ in the above expression, we immediately get
\[
\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i\nu_1 (z_1 - s_1)} \mathcal{W}^D_{f,g}(\frac{s + z}{2}, \omega) e^{-i\nu_2 (z_2 - s_2)} d^2 \omega = f(z) \bar{g}(s). \tag{42}
\]
In particular, if we substitute $s = z = t$ in (42), we obtain
\[
\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{W}^D_{f,g}(t, \omega) d^2 \omega = f(t) \bar{g}(t). \tag{43}
\]


If \( f = g \), then integrating (43) with respect to \( d^2t \) gives the final result
\[
\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{W}_{f}^{D}(t, \omega) \, d^2\omega \, d^2t = \int_{\mathbb{R}^2} |f(t)|^2 \, d^2t. \tag{44}
\]
This gives the desired result. \( \square \)

**Remark 3.2.** Unlike classical case, we cannot establish the frequency energy density, because the Parseval’s formula does not hold for the two-sided QFT.

For an illustrative purpose, we consider an example of the two-sided QWVD.

**Example 3.1.** Given a Gaussian signal defined by
\[
f(t) = e^{-(t_1^2 + t_2^2)},
\]
find the two-sided QWVD of a pure sine wave \( g(t) = e^{j\omega_0 t} e^{j\nu_0 t} \).

From Definition 3.1 we obtain
\[
\mathcal{W}_{f,g}^{D}(t, \omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 t_1} f(t + \frac{\tau_1}{2}) g(t - \frac{\tau_1}{2}) e^{-j\omega_2 \tau_2} \, d^2\tau.
\]
Proposition 3.6. (Shift) Let \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \) be quaternion signals. If we time shift the signals by \( a \), then we obtain
\[
\mathcal{W}_{f,g}^{D}(t, \omega) = \mathcal{W}_{f,g}^{D}(t - a, \omega). \tag{48}
\]
Proposition 3.7. (Nonlinearity) Let \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \) be quaternion signals. The two-sided QWVD is nonlinear. That is
\[
\mathcal{W}_{f+g}^{D}(t, \omega) = \mathcal{W}_{f}^{D}(t, \omega) + \mathcal{W}_{g}^{D}(t, \omega), \tag{49}
\]
Proof. Applying equation (20) and the basic properties of quaternions we get
\[
\mathcal{W}_{f+g}^{D}(t, \omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 t_1} \left( f(t + \frac{\tau_1}{2}) + g(t + \frac{\tau_1}{2}) \right) \left( f(t - \frac{\tau_2}{2}) + g(t - \frac{\tau_2}{2}) \right) e^{-j\omega_2 \tau_2} \, d^2\tau.
\]
\[ \begin{align*}
&= \int_{\mathbb{R}^2} e^{-i\omega_1 t_1} \left( f(t + \frac{\tau}{2}) + g(t + \frac{\tau}{2}) \right) \left( \bar{f}(t - \frac{\tau}{2}) + \bar{g}(t - \frac{\tau}{2}) \right) e^{-j\omega_2 \tau_2} d^2\tau \\
&= \int_{\mathbb{R}^2} e^{-i\omega_1 t_1} \left( f(t + \frac{\tau}{2}) \bar{f}(t - \frac{\tau}{2}) + f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) + g(t + \frac{\tau}{2}) \bar{f}(t - \frac{\tau}{2}) + g(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) \right) e^{-j\omega_2 \tau_2} d^2\tau \\
&\quad + g(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) e^{-j\omega_2 \tau_2} d^2\tau \\
&= \int_{\mathbb{R}^2} e^{-i\omega_1 t_1} f(t + \frac{\tau}{2}) \bar{f}(t - \frac{\tau}{2}) e^{-j\omega_2 \tau_2} d^2\tau + \int_{\mathbb{R}^2} e^{-i\omega_1 t_1} f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) e^{-j\omega_2 \tau_2} d^2\tau \\
&\quad + \int_{\mathbb{R}^2} e^{-i\omega_1 t_1} g(t + \frac{\tau}{2}) \bar{f}(t - \frac{\tau}{2}) e^{-j\omega_2 \tau_2} d^2\tau + \int_{\mathbb{R}^2} e^{-i\omega_1 t_1} g(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) e^{-j\omega_2 \tau_2} d^2\tau \\
&= \mathcal{W}_f^D(t, \omega) + \mathcal{W}_f^D(t, \omega) + \mathcal{W}_f^D(t, \omega) + \mathcal{W}_f^D(t, \omega).
\end{align*} \]

The proof is complete. 

This shows that the two-sided QWVD of the sum of two quaternion signals is not simply the sum of the two-sided QWVD of the signals.

**Proposition 3.8.** (Modulation) Let \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \) be quaternion signals. If we spectrum shift the signal \( f \) by \( \omega_0 \), then we obtain

\[ \begin{align*}
\mathcal{W}_{f, g}^D(t, \omega) &= e^{iu_0 t_1} \mathcal{W}_{f, g_1}^D(t, \omega - \frac{\omega_0}{2}) e^{i\omega_0 t_2} - e^{iu_0 t_1} \mathcal{W}_{f, g_1}^D(t, \omega + \frac{\omega_0}{2}) e^{i\omega_0 t_2} \\
&\quad - e^{iu_0 t_1} \mathcal{W}_{f, g_2}^D(t, \omega - \frac{\omega_0}{2}) j e^{i\omega_0 t_2} - e^{iu_0 t_1} \mathcal{W}_{f, g_2}^D(t, \omega + \frac{\omega_0}{2}) j e^{i\omega_0 t_2} k.
\end{align*} \]

**Proof.** In this proof we shall use the decomposition of quaternion functions. We first remember the fact that \( e^{iu_0 (t_2 + \frac{\omega_0}{2})} \bar{g}_1(t - \frac{\tau}{2}) = \bar{g}_1(t - \frac{\tau}{2}) e^{-j\omega_0 (t_2 + \frac{\omega_0}{2})} \) and so on. This further leads to

\[ \begin{align*}
\mathcal{W}_{f, g}^D(t, \omega) &= \int_{\mathbb{R}^2} e^{-i\omega_1 t_1} \left( \int_{\mathbb{R}^2} e^{-i(\omega_1 - \frac{\omega_0}{2}) t_2} f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 - \frac{\omega_0}{2}) \tau_2} d^2\tau \\
&\quad - \int_{\mathbb{R}^2} e^{-i(\omega_1 + \frac{\omega_0}{2}) t_2} f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 + \frac{\omega_0}{2}) \tau_2} d^2\tau \right) e^{i\omega_0 t_2} \\
&\quad - \int_{\mathbb{R}^2} e^{-i(\omega_1 + \frac{\omega_0}{2}) t_2} \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 + \frac{\omega_0}{2}) \tau_2} d^2\tau \\
&\quad + \int_{\mathbb{R}^2} e^{-i(\omega_1 - \frac{\omega_0}{2}) t_2} \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 - \frac{\omega_0}{2}) \tau_2} d^2\tau \right) e^{i\omega_0 t_2} \\
&\quad - \int_{\mathbb{R}^2} e^{-i(\omega_1 + \frac{\omega_0}{2}) t_2} \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 + \frac{\omega_0}{2}) \tau_2} d^2\tau \\
&\quad + \int_{\mathbb{R}^2} e^{-i(\omega_1 - \frac{\omega_0}{2}) t_2} \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 - \frac{\omega_0}{2}) \tau_2} d^2\tau \right) e^{i\omega_0 t_2} \\
&\quad - \int_{\mathbb{R}^2} e^{-i(\omega_1 + \frac{\omega_0}{2}) t_2} \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 + \frac{\omega_0}{2}) \tau_2} d^2\tau \\
&\quad + \int_{\mathbb{R}^2} e^{-i(\omega_1 - \frac{\omega_0}{2}) t_2} \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 - \frac{\omega_0}{2}) \tau_2} d^2\tau \right) e^{i\omega_0 t_2} \\
&\quad - \int_{\mathbb{R}^2} e^{-i(\omega_1 + \frac{\omega_0}{2}) t_2} \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 + \frac{\omega_0}{2}) \tau_2} d^2\tau \\
&\quad + \int_{\mathbb{R}^2} e^{-i(\omega_1 - \frac{\omega_0}{2}) t_2} \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 - \frac{\omega_0}{2}) \tau_2} d^2\tau \right) e^{i\omega_0 t_2} \\
&\quad - \int_{\mathbb{R}^2} e^{-i(\omega_1 + \frac{\omega_0}{2}) t_2} \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 + \frac{\omega_0}{2}) \tau_2} d^2\tau \\
&\quad + \int_{\mathbb{R}^2} e^{-i(\omega_1 - \frac{\omega_0}{2}) t_2} \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 - \frac{\omega_0}{2}) \tau_2} d^2\tau \right) e^{i\omega_0 t_2} \\
&\quad - \int_{\mathbb{R}^2} e^{-i(\omega_1 + \frac{\omega_0}{2}) t_2} \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 + \frac{\omega_0}{2}) \tau_2} d^2\tau \\
&\quad + \int_{\mathbb{R}^2} e^{-i(\omega_1 - \frac{\omega_0}{2}) t_2} \bar{g}(t - \frac{\tau}{2}) e^{-j(\omega_2 - \frac{\omega_0}{2}) \tau_2} d^2\tau \right) e^{i\omega_0 t_2}.
\end{align*} \]
Applying (56) and Parseval’s formula (14) we get

\[ w_{j,g}^{D}(t,\omega) = \mathcal{F}_{q}^{D}\{\mathcal{F}_{q}^{D}(t,\omega)\}. \tag{57} \]

which completes the proof. \(\square\)

**Proposition 3.9.** (Dilation) Let \( f, g \in L^{2}(\mathbb{R}^{2}; \mathbb{H}) \) be quaternion signals. Then

\[ w_{D,c,q}(t,\omega) = w_{f,g}(t/c, c\omega). \tag{53} \]

**Proof.** Let \( \frac{c}{\omega} = x \), then a direct computation yields

\[ w_{D,c,q}(t,\omega) = \frac{1}{c^{2}} \int_{\mathbb{R}^{2}} e^{-i\omega_{1} \tau_{1}} f\left(\frac{t}{c} + \frac{\tau}{2c}\right) g\left(\frac{t}{c} - \frac{\tau}{2c}\right) e^{-i\omega_{2} \tau_{2}} d^{2}\tau \]

\[ = \int_{\mathbb{R}^{2}} e^{-i\omega_{1} x_{1}} f\left(\frac{t}{c} + \frac{x}{2}\right) g\left(\frac{t}{c} - \frac{x}{2}\right) e^{-i\omega_{2} x_{2}} d^{2}x \]

\[ = w_{f,g}(\frac{t}{c}, c\omega), \tag{54} \]

which was to be proved. \(\square\)

### 3.3 Main Properties of Two-Sided QWVD

In this subsection we investigate two main properties of the two-sided QWVD. Based on the properties of the two-sided QFT we can only establish the specific Moyal’s formula for the QWVD as follows.

**Theorem 3.10** (Moyal’s formula for the two-sided QWVD). Let \( f, g \in L^{2}(\mathbb{R}^{2}; \mathbb{H}) \) be two quaternion signals. Then the following equation holds

\[ \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} |w_{f,g}^{D}(t,\omega)|^{2} d^{2}\omega d^{2}t = (4\pi)^{2} \|f\|_{L^{2}(\mathbb{R}^{2}, \mathbb{H})}^{2} \|g\|_{L^{2}(\mathbb{R}^{2}, \mathbb{H})}^{2}. \tag{55} \]

**Proof.** We first notice that, for fixed \( t \),

\[ w_{f,g}^{D}(t,\omega) = \mathcal{F}_{q}^{D}\{\mathcal{F}_{q}^{D}(t,\omega)\}. \tag{56} \]

Applying (56) and Parseval’s formula (14) we get

\[ \int_{\mathbb{R}^{2}} |w_{f,g}^{D}(t,\omega)|^{2} d^{2}\omega = \|\mathcal{F}_{q}^{D}\{\mathcal{F}_{q}^{D}(t,\omega)\}\|_{L^{2}(\mathbb{R}^{2}; \mathbb{H})}^{2} \]

\[ = (2\pi)^{2} \|\mathcal{F}_{q}^{D}(t,\omega)\|_{L^{2}(\mathbb{R}^{2}; \mathbb{H})}^{2} \]

\[ = (2\pi)^{2} \int_{\mathbb{R}^{2}} f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) \bar{f}(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) d^{2}\tau \]

\[ = (2\pi)^{2} \int_{\mathbb{R}^{2}} f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) g(t - \frac{\tau}{2}) \bar{f}(t + \frac{\tau}{2}) d^{2}\tau. \tag{57} \]

Integrating (57) with respect to \( d^{2}t \) we further obtain

\[ \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} |w_{f,g}^{D}(t,\omega)|^{2} d^{2}\omega d^{2}t = (2\pi)^{2} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) g(t - \frac{\tau}{2}) \bar{f}(t + \frac{\tau}{2}) d^{2}\tau d^{2}t. \tag{58} \]
First use $t + \frac{\tau}{2} = x$, then change the order of integration and use $t - \frac{\tau}{2} = y$ we immediately get

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\mathcal{W}_{f,g}^D(t, \omega)|^2 d^2 \omega d^2 t = (2\pi)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x) \bar{g}(y) g(y) \tilde{f}(x) d^2 y d^2 x$$

$$= (4\pi)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x) \bar{f}(x) d^2 x \int_{\mathbb{R}^2} |g(y)|^2 d^2 y$$

$$= (4\pi)^2 \int_{\mathbb{R}^2} |f(x)|^2 d^2 x \int_{\mathbb{R}^2} |g(y)|^2 d^2 y. \quad (59)$$

This is the desired result. \[ \square \]

Due to the non-commutativity of the quaternion exponential products and quaternion multiplications, we only can establish a special condition of the convolution theorem for the two-sided QWVD (compare to [20]). This is described in the following theorem.

**Theorem 3.11** (Convolution for the two-sided QWVD). Let $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ be two real-valued signals. If we assume that $\mathcal{W}_{f,g}^D$ is a real-valued, then the following result holds:

$$\mathcal{W}_{f*g}^D(t, \omega) = \int_{\mathbb{R}^2} \mathcal{W}_{f}^D(u, \omega) \mathcal{W}_{g}^D(t-u, \omega) d^2 u, \quad (60)$$

where $*$ is the quaternion convolution operator.

**Proof.** Applying the definition of the two-sided QWVD (20) and elementary properties of the quaternion convolution gives

$$\mathcal{W}_{f*g}^D(t, \omega)$$

$$= \int_{\mathbb{R}^2} e^{-i\omega_1 \tau_1} (f * g)(t + \frac{\tau}{2}) (\bar{f} * g)(t - \frac{\tau}{2}) e^{-i\omega_2 \tau_2} d^2 \tau$$

$$= \int_{\mathbb{R}^2} e^{-i\omega_1 \tau_1} (f * g)(t + \frac{\tau}{2}) (\bar{g} * \bar{f})(t - \frac{\tau}{2}) e^{-i\omega_2 \tau_2} d^2 \tau$$

$$= \int_{\mathbb{R}^2} e^{-i\omega_1 \tau_1} \left[ \int_{\mathbb{R}^2} f(x) g(t + \frac{\tau}{2} - x) d^2 x \int_{\mathbb{R}^2} \bar{g}(y) \bar{f}(t - \frac{\tau}{2} - y) d^2 y \right] e^{-i\omega_2 \tau_2} d^2 \tau. \quad (61)$$

By the change of variables $x = u + \frac{p}{2}, y = u - \frac{p}{2}$ and $\tau = p + q$, the above identity can be written as

$$\mathcal{W}_{f*g}^D(t, \omega) = \int_{\mathbb{R}^2} e^{-i(p_1 + q_1)\omega_1} \left[ \int_{\mathbb{R}^2} f(u + \frac{p}{2}) g(t - u + \frac{q}{2}) d^2 x \right.$$

$$\left. \int_{\mathbb{R}^2} \bar{g}(u - \frac{p}{2}) \bar{f}(t - u - \frac{q}{2}) d^2 y \right] e^{-i(p_2 + q_2)\omega_2} d^2 \tau. \quad (62)$$

Because $f, g$ are real-valued signals, we may interchange the order of $g(t - u + \frac{q}{2})$ and $\bar{g}(u - \frac{p}{2})$ to get

$$\mathcal{W}_{f*g}^D(t, \omega) = \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}^2} e^{-i\omega_1} e^{-i\omega_1} f(u + \frac{p}{2}) \bar{g}(u - \frac{p}{2}) e^{-i\omega_2} d^2 p \right.$$

$$\left. \int_{\mathbb{R}^2} g(t - u + \frac{q}{2}) \bar{f}(t - u - \frac{q}{2}) e^{-i\omega_2} d^2 q \right] d^2 u.$$
\[ = \int_{\mathbb{R}^2} \left[ e^{-i\omega_1 \omega_2} W_{f,g}^D(u, \omega) \int_{\mathbb{R}^2} g(t - u + \frac{q}{2}) f(t - u - \frac{q}{2}) e^{-i\omega_2 \omega_2} d^2q \right] d^2u \]

\[ = \int_{\mathbb{R}^2} \left[ W_{f,g}^D(u, \omega) e^{-i\omega_1 \omega_2} \int_{\mathbb{R}^2} g(t - u + \frac{q}{2}) f(t - u - \frac{q}{2}) e^{-i\omega_2 \omega_2} d^2q \right] d^2u \]

\[ = \int_{\mathbb{R}^2} W_{f,g}^D(u, \omega) W_{g,f}^D(t - u, \omega) d^2u, \quad (63) \]

where in the third line we used the assumption to interchange the order of the two-sided QWVD and the kernel of the two-sided QFT. This finishes the proof of the theorem. \( \square \)

4 Right-Sided QWVD

Based on the properties of quaternions and the kernel of the right-sided QFT we may construct the right-sided QWVD. We shall see that some properties of the right-sided QWVD is quite different from the two-sided QWVD.

**Definition 4.1.** The cross right-sided quaternion Wigner-Ville distribution of the 2-D signals \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \) is given by

\[ W_{f,g}^{R}(t, \omega) = \int_{\mathbb{R}^2} f(t + \frac{\tau}{2}) g(t - \frac{\tau}{2}) e^{-i\omega_1 \tau_1} e^{-i\omega_2 \tau_2} d^2\tau, \quad (64) \]

provided the integral exists. \( \square \)

Similar to inverse transform of the two-sided QWVD we get the following fundamental result.

**Theorem 4.1.** The inverse right-sided QWVD of the signal \( f \in L^1(\mathbb{R}^2; \mathbb{H}) \) is given by

\[ f(t) = \frac{1}{(2\pi)^2 \tilde{g}(0)} \int_{\mathbb{R}^2} W_{f,g}^{R}(t, \omega) e^{i\omega_2 \tau_2} e^{i\omega_1 \tau_1} d^2\omega, \quad (65) \]

provided \( \tilde{g}(\mathbf{0}) \neq 0 \).

By using the change of variables \( t + \frac{\tau}{2} = x \), equation (64) can be expressed in the following form

\[ W_{f,g}^{R}(t, \omega) = 4 \int_{\mathbb{R}^2} f(x) \tilde{g}(2t - x) e^{-i\omega_1 (2x - 2t_1)} e^{-i\omega_2 (2x_2 - 2t_2)} d^2x \]

\[ = 4 \int_{\mathbb{R}^2} f(x) \tilde{g}(2t - x) e^{i2\omega_1 t_1} e^{-i2\omega_1 x_1} e^{-i2\omega_2 x_2} d^2x e^{i2\omega_2 t_2}. \quad (66) \]

Observe first that if \( f \) and \( g \) are real signals, the above expression gives an equivalent definition of \( W_{f,g}(t, \omega) \) in the form

\[ W_{f,g}^{R}(t, \omega) = 4 e^{i2\omega_1 t_1} Q_{1g} f(2t, 2\omega) e^{i2\omega_2 t_2}, \quad (67) \]

where \( Q_{1g}(x) = g(-x) \). Here \( Q_{1g} f(2t, 2\omega) \) is the windowed quaternionic Fourier transform which was introduced by Bahri et al. [17, 18].

**Lemma 4.2** (Time energy density). If \( f \in L^2(\mathbb{R}^2; \mathbb{H}) \), then the right-sided QWVD satisfies the time energy density as

\[ \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_{f}^{R}(t, \omega) d^2t d^2\omega = \|f\|^2_{L^2(\mathbb{R}^2; \mathbb{H})}. \quad (68) \]
Proof. Indeed, we have
\begin{align*}
\int_{\mathbb{R}^2} \mathcal{W}_{f,g}^R(t, \omega) \, d^2 \omega &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) e^{-i \omega_1 \tau_1} e^{-i \omega_2 \tau_2} \, d^2 \tau \, d^2 \omega \\
&= (2\pi)^2 \int_{\mathbb{R}^2} f(t + \frac{\tau}{2}) \bar{g}(t - \frac{\tau}{2}) \delta(-\tau) \, d^2 \tau \\
&= (2\pi)^2 f(t) \bar{g}(t).
\end{align*}
(69)

Integrating both sides of the above expression with respect to \( t \), we obtain
\begin{align*}
\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{W}_{f,g}^R(t, \omega) \, d^2 t \, d^2 \omega &= \int_{\mathbb{R}^2} f(t) \bar{g}(t) \, d^2 t.
\end{align*}
(70)

In particular, \( f = g \), we get
\begin{align*}
\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{W}_{f,f}^R(t, \omega) \, d^2 \omega \, d^2 t &= \int_{\mathbb{R}^2} |f(t)|^2 \, d^2 t.
\end{align*}
(71)

This gives the desired result. \( \square \)

4.1 Useful Properties

Proceeding as in the proof of Propositions listed in Section 3.2 we obtain elementary properties of the right-sided QWVD.

Proposition 4.3. For \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \) the right-sided QWVD has the following useful properties.

(i). (Shift)
\begin{align*}
\mathcal{W}_{f,g}^R(T_a f, T_a g)(t, \omega) &= \mathcal{W}_{f,g}^R(t - a, \omega).
\end{align*}
(72)

(ii). (Nonlinearity)
\begin{align*}
\mathcal{W}_{f+g}^R(t, \omega) &= \mathcal{W}_f^R(t, \omega) + \mathcal{W}_{f,g}(t, \omega) + \mathcal{W}_{g,f}(t, \omega) + \mathcal{W}_g(t, \omega).
\end{align*}
(73)

(iii). (Dilation)
\begin{align*}
\mathcal{W}_{D_c f, D_c g}^R(t, \omega) &= \mathcal{W}_{f,g}^R\left(\frac{t}{c}, c\omega\right).
\end{align*}
(74)

Remark 4.1. It seems that modulation property is not valid for the right-sided QWVD. This shows that some properties of the right-sided QWVD follows the properties the right-sided QFT.

In the following we establish general Moyal’s formula of the right-sided QWVD. We see that Moyal’s formula of the two-sided QWVD is a special case of Moyal’s formula of the right-sided QWVD

Theorem 4.4 (Moyal’s formula for right-sided the QWVD). Let \( f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^2; \mathbb{H}) \) be quaternion-valued signals. Then the following equation holds
\begin{align*}
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{W}_{f_1,g_1}^R(t, \omega) \mathcal{W}_{f_2,g_2}^R(t, \omega) \, d^2 t \, d^2 \omega &= (4\pi)^2 (f_1(\bar{g}_1, \bar{g}_2)_{L^2(\mathbb{R}^2; \mathbb{H})}, f_2)_{L^2(\mathbb{R}^2; \mathbb{H})}.
\end{align*}
(75)
Proof. From the definition of the right-sided QWVD (64), we easily obtain

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_{f_1,g_1}^R(t,\omega) W_{f_2,g_2}^R(t,\omega) \, d^2t \, d^2\omega
\]

\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_1(t + \frac{\tau}{2}) g_1(t - \frac{\tau}{2}) e^{i\omega_1(\tau_1' - \tau_1)} e^{-i\omega_2(\tau_2' - \tau_2)}
\times g_2(t - \frac{\tau'}{2}) \bar{f}_2(t + \frac{\tau'}{2}) \, d^2\tau' \, d^2\tau \, d^2t \, d^2\omega
\]

\[
= (2\pi)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_1(t + \frac{\tau}{2}) g_1(t - \frac{\tau}{2}) \delta(\tau_1' - \tau_1) \delta(\tau_2' - \tau_2) g_2(t - \frac{\tau'}{2}) \bar{f}_2(t + \frac{\tau'}{2}) \, d^2\tau' \, d^2\tau \, d^2t
\]

\[
= (2\pi)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_1(t + \frac{\tau}{2}) g_1(t - \frac{\tau}{2}) g_2(t - \frac{\tau}{2}) \bar{f}_2(t + \frac{\tau}{2}) \, d^2\tau \, d^2t.
\]

Letting \( t + \frac{\tau}{2} = x, d\tau = 4dx \) and putting \( t - \frac{\tau}{2} = y, dt = dy \)

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_{f_1,g_1}^R(t,\omega) W_{f_2,g_2}^R(t,\omega) \, d^2t \, d^2\omega = (2\pi)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_1(x) g_1(y) g_2(y) \bar{f}_2(x) \, d^2x \, d^2y.
\]

Interchanging the order of integration gives

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_{f_1,g_1}^R(t,\omega) W_{f_2,g_2}^R(t,\omega) \, d^2t \, d^2\omega = (4\pi)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_1(x) g_1(y) g_2(y) \bar{f}_2(x) \, d^2x \, d^2y
\]

\[
= (4\pi)^2 (f_1(\bar{g}_1, \bar{g}_2) L^2(\mathbb{R}^2; \mathbb{H}), f_2) L^2(\mathbb{R}^2; \mathbb{H}).
\]

This completes the proof of the theorem. \( \square \)

Based on the above theorem, we may conclude the following important consequences.

(i). If \( g_1 = g_2 \), then

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_{f_1,g_1}^R(t,\omega) W_{f_2,g_2}^R(t,\omega) \, d^2t \, d^2\omega = (4\pi)^2 \| g_1 \|^2_{L^2(\mathbb{R}^2; \mathbb{H})} (f_1, f_2)_{L^2(\mathbb{R}^2; \mathbb{H})}.
\] (76)

This formula is quite similar to the Moyal’s formula for the classical WVD, for example, see [5] and [10]. However, we must remember that equation (76) is a quaternion valued function.

(ii). If \( f_1 = f_2 \), then

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W_{f_1,g_1}^R(t,\omega) W_{f_1,g_2}^R(t,\omega) \, d^2t \, d^2\omega = (4\pi)^2 (f_1(\bar{g}_1, \bar{g}_2) L^2(\mathbb{R}^2; \mathbb{H}), f_1)_{L^2(\mathbb{R}^2; \mathbb{H})}.
\] (77)

(iii). If \( f_1 = f_2 \) and \( g_1 = g_2 \), then

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \| W_{f_1,g_1}^R(t,\omega) \|^2 \, d^2t \, d^2\omega = (4\pi)^2 \| f_1 \|^2_{L^2(\mathbb{R}^2; \mathbb{H})} \| g_1 \|^2_{L^2(\mathbb{R}^2; \mathbb{H})}.
\] (78)

This formula has the same form as the specific Moyal’s formula for the two-sided QWVD (55) and also the classical WVD.
5 Quaternion Ambiguity Function (QAF)

The classical ambiguity function (AF) is firstly introduced by Woodward in 1953 for mathematical analysis of sonar and radar signals [5]. This section shall generalize the classical AF in the quaternion algebra setting.

**Definition 5.1.** The cross two-sided quaternionic ambiguity function (QAF) of the two-dimensional functions (or signals) \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \) is denoted by \( A_{f,g}(t, \omega) \) and is defined by

\[
A_{f,g}^D(t, \omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 \tau_1} f(\tau + \frac{t}{2}) \overline{g}(\tau - \frac{t}{2}) e^{-i\omega_2 \tau_2} d^2\tau, \tag{79}
\]

provided the integral exists.

The following lemma describes the relationship between the two-sided QWVD and the two-sided QAF mentioned above.

**Lemma 5.1.** The two-sided QWVD of the signal \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \) can be seen as the two-sided QAF of the signals \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \) by formula

\[
A_{f,g}^D(2t, 2\omega) = \frac{1}{4} \mathcal{W}_{f,g}(t, \omega). \tag{80}
\]

**Proof.** Putting \( \tau + \frac{t}{2} = x \), we may write equation (79) in the form

\[
A_{f,g}^D(t, \omega) = \int_{\mathbb{R}^2} e^{-i\omega_1(x_1 - \frac{t}{2})} f(x) \overline{g}(x - t)e^{-i\omega_2(x_2 - \frac{t}{2})} d^2x = e^{i\omega_1 t_1} \int_{\mathbb{R}^2} e^{-i\omega_1 x_1} f(x) \overline{g}(x - t)e^{-i\omega_2 x_2} d^2x e^{i\omega_2 t_2}. \tag{81}
\]

It means that we have

\[
4A_{f,g}^D(2t, 2\omega) = 4e^{i2\omega_1 t_1} \int_{\mathbb{R}^2} e^{-i2\omega_1 x_1} f(x) \overline{g}(- (x - 2t))e^{-i2\omega_2 x_2} d^2x e^{i2\omega_2 t_2} = \mathcal{W}_{f,g}(t, \omega). \tag{82}
\]

We should remember that if we write \( h_\tau(t) = f(\tau + \frac{t}{2}) \overline{g}(\tau - \frac{t}{2}) \), then the two-sided QWVD defined in (79) is the two-sided QFT of the function \( h_\tau(t) \) with respect to \( t \). That is

\[
A_{f,g}^D(t, \omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 \tau_1} h_\tau(t)e^{-i\omega_2 \tau_2} d^2\tau = \mathcal{F}^D_q \{ h_\tau(t) \}(\omega), \tag{83}
\]

which was to be proved. \( \square \)

We also obtain the following results which correspond to classical WVD properties (compare to [5, 6]).

**Lemma 5.2.** For \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \), the two-sided QFT of (79) with respect to \( \omega \) can be represented in the form

\[
\mathcal{F}_q^D \{ A_{f,g}^D \}(t, \sigma) = \int_{\mathbb{R}^2} e^{-i\omega_1 \sigma_1} A_{f,g}^D(t, \omega) e^{-i\omega_2 \sigma_2} d^2\omega = (2\pi)^2 f(-\sigma + \frac{t}{2}) \overline{g}(-\sigma - \frac{t}{2}). \tag{84}
\]
Proof. Direct calculations yield
\[ F_q^D \{ A_{f,g}^D \}(t, \sigma) \equiv \int_{\mathbb{R}^2} e^{-i\omega_1 \tau_1} A_{f,g}(t, \omega) e^{-j\omega_2 \tau_2} d^2 \omega \]
\[ = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-i\omega_1 \tau_1} h_t(\tau)e^{-j\omega_2 \tau_2} d^2 \tau e^{-j\omega_2 \tau_2} d^2 \omega \]
\[ = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i\omega_1 (-\tau_1 - \sigma_1)} h_t(\tau) e^{j\omega_2 (-\tau_2 - \sigma_2)} d^2 \omega d^2 \tau \]
\[ = (2\pi)^2 h_t(-\sigma) = (2\pi)^2 f(-\sigma + t/2) \bar{g}(-\sigma - t/2), \] (85)
which was to be proved. \( \square \)

The following theorem shows that the quaternion signal can be recovered from the two-sided QAF up to a quaternion constant.

Theorem 5.3 (Reconstruction formula for two-sided QAF). The inverse transform of the cross two-sided QWVD of the signal \( f \in L^2(\mathbb{R}^2; \mathbb{H}) \) is given by
\[ f(t) = \frac{1}{(2\pi)^2 \bar{g}(0)} \int_{\mathbb{R}^2} e^{i\omega_1 \tau_1} A_{f,g}^D(t, \omega) e^{j\omega_2 \tau_2} d^2 \omega, \] (86)
provided \( \bar{g}(0) \neq 0 \).

Proof. We have from the inverse transform of the two-sided QFT (12)
\[ f(\tau + t/2) \bar{g}(\tau - t/2) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\omega_1 \tau_1} A_{f,g}^D(t, \omega) e^{j\omega_2 \tau_2} d^2 \omega. \] (87)
Taking the specific value, \( \tau = \frac{t}{2} \), we have
\[ f(t) \bar{g}(0) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\omega_1 \tau_1} A_{f,g}^D(t, \omega) e^{j\omega_2 \tau_2} d^2 \omega. \] (88)
Or, equivalently,
\[ f(t) = \frac{1}{(2\pi)^2 \bar{g}(0)} \int_{\mathbb{R}^2} e^{i\omega_1 \tau_1} A_{f,g}^D(t, \omega) e^{j\omega_2 \tau_2} d^2 \omega, \] (89)
which completes the proof. \( \square \)

5.1 Useful Properties of Two-Sided QAF

The properties of the two-sided QAF are summarized in the following proposition. It seem that they have a remarkable similarity with those of the two-sided QWVD.

Proposition 5.4. For \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \) the two-sided QAF has the following important properties.

(i). (Shift)
\[ A_{T_{a_1}f, T_{a_2}g}(t, \omega) = e^{-i\omega_1 a_1} A_{f,g}^D(t, \omega) e^{-j\omega_2 a_2}. \] (90)

(ii). (Nonlinearity)
\[ A_{f+g}^D(t, \omega) = A_f^D(t, \omega) + A_{f,g}^D(t, \omega) + A_{g,f}^D(t, \omega) + A_g^D(t, \omega). \] (91)
(iii). (Modulation)

\[
A_{M_{\omega_0}, f,g}^D(t, \omega) = e^{in_1 \tau_1} A_{D_{\omega_0}}^D(t, \omega - \frac{\omega_0}{2}) e^{j\omega_1 t} - e^{in_1 \tau_1} A_{f,g_1}^D(t, \omega_1 - \frac{u_0}{2}, \omega_2 + \frac{v_0}{2}) e^{-j\omega_2 t} \bar{i}
\]

\[
- e^{in_1 \tau_1} A_{f,g_2}^D(t, \omega - \frac{\omega_0}{2}) e^{j\omega_2 t} \bar{j} - e^{in_1 \tau_1} A_{f,g_3}^D(t, \omega_1 - \frac{u_0}{2}, \omega_2 + \frac{v_0}{2}) e^{-j\omega_2 t} \bar{k}.
\]

(92)

(iv). (Dilation)

\[
A_{D_{c, D_{c,g}}}^D(t, \omega) = A_{f,g}^D(\frac{t}{c}, c\omega).
\]

(93)

Proof. For part (i), using the definition of the two-sided QAF (79) we obtain

\[
A_{T_{a,f}, T_{a,g}}(t, \omega)
= \int_{\mathbb{R}^2} e^{-i\omega_1 \tau_1} f(\tau - a + \frac{t}{2}) \bar{g}(\tau - a - \frac{t}{2}) e^{-j\omega_2 \tau_2} d^2 \tau
\]

\[
= \int_{\mathbb{R}^2} e^{-i\omega_1 (\tau_1 - a + a_1)} f(\tau - a + \frac{t}{2}) \bar{g}(\tau - a - \frac{t}{2}) e^{-j\omega_2 (\tau_2 - a_2 + a_2)} d^2 \tau
\]

\[
eq e^{-i\omega_1 a_1} \int_{\mathbb{R}^2} e^{-i\omega_1 (\tau_1 - a_1)} f(\tau - a + \frac{t}{2}) \bar{g}(\tau - a - \frac{t}{2}) e^{-j\omega_2 (\tau_2 - a_2)} d^2 \tau e^{-j\omega_2 a_2}
\]

\[
eq e^{-i\omega_1 a_1} A_{f,g}^D(t, \omega) e^{-j\omega_2 a_2}.
\]

(94)

To derive part (iii), we apply equation (79) and the decomposition of quaternion function to get

\[
A_{M_{\omega_0}, f,g}^D(t, \omega)
= \int_{\mathbb{R}^2} e^{-i\omega_1 \tau_1} \left( e^{in_1 (\tau_1 + \frac{\tau}{2})} f(\tau + \frac{t}{2}) e^{j\omega_1 (\tau_2 + \frac{\tau}{2})} \bar{g}(\tau - \frac{t}{2}) \right) e^{-j\omega_2 \tau_2} d^2 \tau
\]

\[
= \int_{\mathbb{R}^2} e^{in_1 \tau_1} \left( e^{-i(\omega_1 - u_0) \tau_1} f(\tau + \frac{t}{2}) e^{j\omega_1 (\tau_2 + \frac{\tau}{2})} \bar{g}(\tau - \frac{t}{2}) - i\bar{g}_1(\tau - \frac{t}{2}) - i\bar{g}_2(\tau - \frac{t}{2})
\]

\[-k\bar{g}_3(\tau - \frac{t}{2}) \right) e^{-j\omega_2 \tau_2} d^2 \tau
\]

\[
eq e^{in_1 \tau_1} \left( \int_{\mathbb{R}^2} e^{-i(\omega_1 - u_0) \tau_1} f(\tau + \frac{t}{2}) \bar{g}_0(\tau - \frac{t}{2}) e^{-j(\omega_2 - v_0) \tau_2} d^2 \tau
\]

\[- \int_{\mathbb{R}^2} e^{-i(\omega_1 - u_0) \tau_1} f(\tau + \frac{t}{2}) \bar{g}_2(\tau - \frac{t}{2}) e^{-j(\omega_2 - v_0) \tau_2} d^2 \tau \right) e^{j\omega_1 \tau_2} \bar{j}
\]

\[- e^{in_1 \tau_1} \left( \int_{\mathbb{R}^2} e^{-i(\omega_1 - u_0) \tau_1} f(\tau + \frac{t}{2}) \bar{g}_1(\tau - \frac{t}{2}) e^{j(\omega_2 + v_0) \tau_2} d^2 \tau \right) e^{-j\omega_1 \tau_2} \bar{i}
\]

\[- e^{in_1 \tau_1} \left( \int_{\mathbb{R}^2} e^{-i(\omega_1 - u_0) \tau_1} f(\tau + \frac{t}{2}) \bar{g}_3(\tau - \frac{t}{2}) e^{j(\omega_2 + v_0) \tau_2} d^2 \tau \right) e^{-j\omega_1 \tau_2} \bar{k}
\]

\[
eq e^{in_1 \tau_1} \left( \int_{\mathbb{R}^2} e^{-i(\omega_1 - u_0) \tau_1} f(\tau + \frac{t}{2}) \bar{g}_0(\tau - \frac{t}{2}) e^{-j(\omega_2 - v_0) \tau_2} d^2 \tau e^{j\omega_1 \tau_2}
\]

\[- e^{in_1 \tau_1} \left( \int_{\mathbb{R}^2} e^{-i(\omega_1 - u_0) \tau_1} f(\tau + \frac{t}{2}) \bar{g}_1(\tau - \frac{t}{2}) e^{j(\omega_2 + v_0) \tau_2} d^2 \tau e^{-j\omega_1 \tau_2} \bar{i}
\]

\[- e^{in_1 \tau_1} \left( \int_{\mathbb{R}^2} e^{-i(\omega_1 - u_0) \tau_1} f(\tau + \frac{t}{2}) \bar{g}_2(\tau - \frac{t}{2}) e^{-j(\omega_2 - v_0) \tau_2} d^2 \tau e^{j\omega_1 \tau_2} \bar{j}
\]

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\[-e^{iu_{\frac{1}{2}}t} \int_{\mathbb{R}^2} e^{-i(\omega_1 - \frac{u_2}{2})t} f(\tau + \frac{t}{2})g(\tau - \frac{t}{2})e^{j(\omega_2 + \frac{u_2}{2})}\sigma} d^2\sigma e^{-j\frac{u_2}{2}k} = e^{iu_{\frac{1}{2}}t} A_{fg}(t, \omega - \frac{\omega_0}{2})e^{iu_{\frac{1}{2}}t} - e^{iu_{\frac{1}{2}}t} A_{fg}(t, \omega - \frac{\omega_0}{2})e^{iu_{\frac{1}{2}}t}j - e^{iu_{\frac{1}{2}}t} A_{fg}(t, \omega - \frac{\omega_0}{2})e^{iu_{\frac{1}{2}}t} \bar{g}(\omega) e^{j(\omega_2 + \frac{u_2}{2})}\sigma} d^2\sigma. \] 

For part (iv), let $\frac{\sigma}{c} = x$, we have, by definition,

\[
A_{D_{cf},D_{cs}}(t, \omega) = \frac{1}{c} \int_{\mathbb{R}^2} e^{-i\omega_1x} f(\frac{t}{c} + \frac{x}{2}) \bar{g}(\frac{t}{c} - \frac{x}{2}) e^{-j\omega_2x} d^2x = \int_{\mathbb{R}^2} e^{-i\omega_1x} f(\frac{t}{c} + \frac{x}{2}) \bar{g}(\frac{t}{c} - \frac{x}{2}) e^{-j\omega_2x} d^2x = A_{D_{fg}}(\frac{t}{c}, \omega). \tag{96}
\]

**Theorem 5.5** (Specific Moyal’s formula for the two-sided QAF). Suppose that $f \in L^2(\mathbb{R}^2; \mathbb{H})$ and $g \in L^2(\mathbb{R}^2; \mathbb{H})$ are two quaternion signals. Then the following equation holds

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |A_{D_{fg}}(t, \omega)|^2 d^2\omega d^2t = (4\pi)^2 \|f\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2 \|g\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2. \tag{97}
\]

**Proof.** Using Plancherel’s formula for the right-sided QFT in equation (14) and then applying Lemma 5.2 we immediately obtain

\[
\int_{\mathbb{R}^2} |A_{D_{fg}}(t, \omega)|^2 d^2\omega = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\mathcal{F}_q \{A_{D_{fg}}\}(t, \sigma)|^2 d^2\sigma = (2\pi)^2 \int_{\mathbb{R}^2} f(-\sigma + \frac{t}{2}) \bar{g}(-\sigma - \frac{t}{2}) f(-\sigma + \frac{t}{2}) \bar{g}(-\sigma - \frac{t}{2}) d^2\sigma = (2\pi)^2 \int_{\mathbb{R}^2} f(-\sigma + \frac{t}{2}) \bar{g}(-\sigma - \frac{t}{2}) f(-\sigma + \frac{t}{2}) \bar{g}(-\sigma - \frac{t}{2}) d^2\sigma. \tag{98}
\]

Integrating (98) with respect to $d^2t$ yields

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |A_{D_{fg}}(t, \omega)|^2 d^2\omega d^2t = (2\pi)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(-\sigma + \frac{t}{2}) \bar{g}(-\sigma - \frac{t}{2}) f(-\sigma + \frac{t}{2}) \bar{g}(-\sigma - \frac{t}{2}) d^2\sigma d^2t.
\]

Putting $-\sigma + \frac{t}{2} = x$ and $-\sigma - \frac{t}{2} = y$, then the right side of the above expression can be rewritten as

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |A_{D_{fg}}(t, \omega)|^2 d^2\omega d^2t = (4\pi)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x) \bar{g}(y) f(x) \bar{g}(y) f(x) d^2y d^2x = (4\pi)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x) |g(y)|^2 f(x) d^2y d^2x. \tag{99}
\]

After switching the order of integration, we get

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |A_{D_{fg}}(t, \omega)|^2 d^2\omega d^2t = (4\pi)^2 \|f\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2 \|g\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2.
\]

This is the desired result. \qed
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References


Conflict of Interests

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