Prime factor rings of skew polynomial rings over a commutative Dedekind domain

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Abstract

This paper is concerned with prime factor rings of a skew polynomial ring over a commutative Dedekind domain. Let $P$ be a non-zero prime ideal of a skew polynomial ring $R = D[x; \sigma]$, where $D$ is a commutative Dedekind domain and $\sigma$ is an automorphism of $D$. If $P$ is not a minimal prime ideal of $R$, then $R/P$ is a simple Artinian ring. If $P$ is a minimal prime ideal of $R$, then there are two different types of $P$, namely, either $P = p[x; \sigma]$ or $P = P' \cap R$, where $p$ is a $\sigma$-prime ideal of $D$, $P'$ is a prime ideal of $K[x; \sigma]$ and $K$ is the quotient field of $D$. In the first case $R/P$ is a hereditary prime ring and in the second case, it is shown that $R/P$ is a hereditary prime ring if and only if $M^2 \nsubseteq P$ for any maximal ideal $M$ of $R$. We give some examples of minimal prime ideals such that the factor rings are not hereditary or hereditary or Dedekind, respectively.

Keywords: minimal prime, prime factor, hereditary, Dedekind domain.

0 Introduction
Let $D$ be a commutative Dedekind domain with its quotient field $K$ and let $\sigma$ be an automorphism of $D$. We denote by $R = D[x; \sigma]$ the skew polynomial ring over $D$ in an indeterminate $x$.

The aim of the paper is to study the structure of the prime factor ring $R/P$ for any prime ideal $P$ of $R$, which is one of the ways to investigate the structure of rings. If $P$ is not a minimal prime ideal of $R$, then the Krull dimension of $R/P$ is zero (see [MR]), that is, it is a simple Artinian ring. So we can restrict to the case $P$ is a minimal prime ideal of $R$. There are two types of minimal prime ideals $P$ of $R$, that is, either $P = p[x; \sigma]$ or $P = P' \cap R$, where $p$ is a non-zero $\sigma$-prime ideal of $D$ and $P'$ is a non-zero prime ideal of $K[x; \sigma]$. In the first case $R/P$ is always a hereditary prime ring. In the second case $R/P$ is a hereditary prime ring if and only if $P \not\subseteq M^2$ for any maximal ideal $M$ of $R$, which is motivated by [H] and he only considered in the case where $P$ is principal generated by a monic polynomial and $\sigma = 1$ (note that in this case, $P$ is a minimal prime ideal and see [PR] and [MLP] for related papers). We give some examples of minimal prime ideals $P$ such that $R/P$ is not hereditary or hereditary or Dedekind, respectively, by using Gauss’s integers $D = \mathbb{Z} \oplus \mathbb{Z}i$, where $\mathbb{Z}$ is the ring of integers.

We refer the readers to [MR] and [MMU] for some known terminologies not defined in this paper.

1 Notes on hereditary prime PI rings

Throughout this section, let $R$ be a hereditary prime PI ring with the center $C$ and let $Q$ be the quotient ring of $R$, which is a simple Artinian ring. It is well known that $R$ is a classical $C$-order in $Q$ and that $C$ is a Dedekind domain (see [MR, (13.9.16)]).

In this section, we will shortly discuss some relations between the maximal ideals of $R$ and $C$, which are used in latter sections. For any $R$-ideal $A$, we use the following notation:

$$(R : A)_l = \{ q \in Q \mid QA \subseteq R \}, \quad (R : A)_r = \{ q \in Q \mid Aq \subseteq R \},$$

$$(A : A)_l = \{ q \in Q \mid QA \subseteq A \} = O_l(A), \text{ the left order of } A,$$

$$(A : A)_r = \{ q \in Q \mid Aq \subseteq A \} = O_r(A), \text{ the right order of } A,$$

and

$$A_v = (R : (R : A)_l)_r, \quad vA = (R : (R : A)_r)_l,$$

which are both $R$-ideals containing $A$. Note that $A_v = A = vA$, because $R$ is a hereditary prime ring. A finite set of distinct idempotent maximal ideals $M_1, \ldots, M_m$ of $R$ such that $O_r(M_1) = O_l(M_2), \ldots, O_r(M_m) = O_l(M_1)$ is called a cycle. We will also consider an invertible maximal ideal to be a trivial case of a cycle.

It is well known that an ideal $P$ is a maximal invertible ideal if and only if $P = M_1 \cap \ldots \cap M_m$, where $M_1, \ldots, M_m$ is a cycle (see [ER, (2.5) and (2.6)]). Let $P$ be a maximal invertible ideal. Then $C(P) = \{ c \in R \mid c \text{ is regular mod } P \}$ is a regular Ore set and we denote by $R_P$ the localization of $R$ at $P$ (see [M_1, proposition 2.7]). We denote by $\text{Spec}(R)$ and $\text{Max-in}(R)$ the set of all prime ideals and the set of all maximal invertible ideals, respectively. For any ring $S$, $J(S)$ stands for Jacobson
radical of $S$.

**Lemma 1.1.** (1) Let $P \in \text{Max-in}(R)$ and let $p = P \cap C$. Then $p \in \text{Spec}(C)$.

(2) $C$ is a discrete rank one valuation ring if and only if $J(R)$ of $R$ is the intersection of a cycle.

**Proof.** (1) Let $P = M_1 \cap \ldots \cap M_m \in \text{Max-in}(R)$. If $m = 1$, then $p = P \cap C \in \text{Spec}(C)$. If $m \geq 2$, then $M_i$ are all idempotents. Set $p = M_1 \cap C$, then $M_1 \supseteq pR$, an invertible ideal. So

$$(R : M_2)_l = O_l(M_2) = O_r(M_1) = (R : M_1)_r \subseteq (R : pR)_r = (R : pR)_l$$

imply

$$M_2 = (M_2)_v = (R : (R : M_2)_l) \supseteq (R : (R : pR)_l)_r = pR.$$ 

Thus $M_2 \cap C = p$ follows. Continuing this process, we have $P \cap C = p$.

(2) Suppose that $C$ is a discrete rank one valuation ring with $J(C) = p$, the unique maximal ideal. Then $J(R) \supseteq pR$ (see [R, (6.15)]). So $J(R)$ is invertible by [ER, (4.13)]. Let $J(R) = P_1 \cap \ldots \cap P_k$, where $P_i \in \text{Max-in}(R)$. It suffices to prove that $k = 1$. We assume that $k \geq 2$. Then $P_1 \supset R$ and $Z(R_{P_1}) \supseteq Z(R) = C$, where $Z(R_{P_1})$ is the center of $R_{P_1}$, so that $Z(R_{P_1}) = C$. Since $R_{P_1}$ is a finitely generated $C$-module (see [MR, (13.9.16)]), there is a $c \in C(P_1)$ with $R_{P_1} = cR_{P_1} \subseteq R$, a contradiction. Hence $k = 1$ and so $J(R)$ is the intersection of a cycle.

Suppose that $J(R)$ is the intersection of a cycle. Then $p = J(R) \cap C \in \text{Spec}(C)$ by (1). Let $P_1 \in \text{Spec}(C)$. Then $P_1R = J(R)^l$ for some $l \geq 1$ by [ER, (2.1)] and the assumption. It follows that $P_1 \subseteq J(R) \cap C = p$ and so $P_1 = p$, that is, $C$ is a discrete rank one valuation ring.

The following proposition is just a generalization of a Dedekind $C$-order to a hereditary prime PI ring (see, [R, (22.4)]).

**Proposition 1.2.** Suppose that $R$ is a hereditary prime PI ring. Then there is a one-to-one correspondence between Max-in($R$) and Spec($C$), which is given by: $P \longrightarrow p = P \cap C$, where $P \in \text{Max-in}(R)$.

**Proof.** Let $P \in \text{Max-in}(R)$. Then $p = P \cap C \in \text{Spec}(C)$ by Lemma 1.1. Conversely, let $p \in \text{Spec}(C)$. Then there is a maximal ideal $M$ of $R$ containing $pR$, an invertible ideal. So there is a $P \in \text{Max-in}(R)$ with $P \supseteq pR$ by [ER, (2.4)]. This shows $P \cap C = p$ by lemma 1.1. To prove the correspondence is one-to-one, let $P, P_1 \in \text{Max-in}(R)$ with $P \cap C = p = P_1 \cap C$. Then $P_p, P_{1p} \in \text{Max-in}(R_p)$ and $Z(R_p) = C_p$, a discrete rank one valuation ring. Thus $P_p = J(R_p) = P_{1p}$ by lemma 1.1 and so $P = P_p \cap R = P_{1p} \cap R = P_1$. Hence the correspondence is one-to-one.

## 2 Prime factor rings of skew polynomial rings

Throughout this section, let $D$ be a commutative Dedekind domain with its quotient field $K$ and $\sigma$ be an automorphism of $D$. We always assume that $D \neq K$ to avoid the
trivial case. Let $R = D[x; \sigma]$, a skew polynomial ring over $D$.

The aim of this section is to study the structure of the factor rings of $R$ by minimal prime ideals. It is well known that $R$ is a Noetherian maximal order in $K(x; \sigma)$, the quotient ring of $K[x; \sigma]$ and $\text{gl.dim } R = 2$ (see [C. Proposition 3.3] and [MR, (7.5.3)]). We denote by $\text{Spec}_0(R) = \{ P \in \text{Spec}(R) \mid P \cap D = (0) \}$. It is well known that there is a one-to-one correspondence between $\text{Spec}_0(R)$ and $\text{Spec}(K[x; \sigma])$, which is given by $P \rightarrow P' = PK[x; \sigma]$ and $P' \rightarrow P' \cap R$, where $P \in \text{Spec}_0(R)$ and $P' \in \text{Spec}(K[x; \sigma])$ (see [GW, (9.22)]).

We start with the following easy proposition.

**Proposition 2.1.** (1) $\{ p[x; \sigma], \ P \mid p \text{ is a } \sigma\text{-prime ideal of } D \text{ and } P \in \text{Spec}_0(R) \text{ with } P \neq (0) \}$ is the set of all minimal prime ideals of $R$.

(2) Let $P \in \text{Spec}(R)$ with $P \neq (0)$. Then $P$ is invertible if and only if it is a minimal prime ideal of $R$.

**Proof.** (1) Let $P$ be a minimal prime ideal of $R$ and let $p = P \cap D$. If $p = (0)$, then $P \in \text{Spec}_0(R)$. If $p \neq (0)$, then there are two cases: namely, either $x \in P$ or $x \notin P$. Suppose that $x \in P$. Then $P = p + xR \supset xR$, a prime ideal, which is a contradiction. So $x \notin P$. Then $p$ is a $\sigma$-prime ideal of $D$ and $p[x; \sigma]$ is a prime ideal of $R$. Hence $P = p[x; \sigma]$ follows.

Conversely, let $P \in \text{Spec}_0(R)$. Then $P$ is a minimal prime ideal of $R$, because $P' = PK[x; \sigma]$ is a maximal ideal as well as a minimal prime ideal of $K[x; \sigma]$. Let $P = p[x; \sigma]$, where $p$ is a $\sigma$-prime ideal. Then $P$ is invertible, because $p$ is invertible and so $P$ is a $v$-ideal. Hence $P$ is a minimal prime ideal of $R$ (see [MR, (5.1.9)])

(2) Let $P$ be a prime and invertible ideal. Then it is a $v$-ideal and so it is a minimal prime ideal (see [MR, (5.1.9)]).

Conversely, let $P$ be a minimal prime ideal. If $P = p[x; \sigma]$, where $p$ is a $\sigma$-prime ideal of $D$. Then $P$ is invertible. If $P \in \text{Spec}_0(R)$, with $P \neq (0)$ and $P' = PK[x; \sigma]$, then since any ideal of $K[x; \sigma]$ is a $v$-ideal and $R$ is Noetherian, we have

$$P' = P_\sigma = (K[x; \sigma] : (K[x; \sigma] : P')_R) = (K[x; \sigma] : K[x; \sigma](R : P)_R) = (R : (R : P)_R)K[x; \sigma] = P, K[x; \sigma].$$

Thus $P = P' \cap R = P_\sigma$ follows and similarly $P = vP$. Hence $P$ is invertible by [CS, p.324].

**Proposition 2.2.** (1) Let $P$ be a minimal prime ideal of $R$ with $P = p[x; \sigma]$, where $p$ is a $\sigma$-prime ideal of $D$. Then $R/P$ is a hereditary prime ring. In particular, $R/P$ is a Dedekind prime ring if and only if $p \in \text{Spec}(D)$.

(2) Suppose that $\sigma$ is of infinite order. Then $P = xR$ is the only minimal prime ideal of $R$ in $\text{Spec}_0(R)$ and $R/P$ is a Dedekind prime ring.

**Proof.** (1) The first statement follows from [MR, (7.5.3)]. If $p \in \text{Spec}(D)$. Then $(R/P) \cong (D/p)[x; \sigma]$ is a principal ideal ring so that $R/P$ is a Dedekind prime ring. If $p \notin \text{Spec}(D)$, then there is a maximal ideal $m$ of $D$ with $m \supset p$ and $p = m \cap \sigma(m) \cap \ldots \cap \sigma^n(m)$ for some natural number $n \geq 1$. Set $M = m + xR$, a
maximal ideal of $R$. Then $M = M^2 + P$, because $m^2 + p = m$. Thus $M/P$ is idempotent and $R/P$ is not Dedekind.

(2) Let $P = xR$. Then $P$ is the only minimal prime ideal of $R$ in $\text{Spec}_0(R)$ by [J, Theorem 2] and $R/P$ is a Dedekind prime ring because $(R/P) \cong D$.

Because of Propositions 2.1 and 2.2, we may assume that $\sigma$ is of finite order to study the hereditaryness of $R/P$. So in the remainder of this section, we may assume that $\sigma$ is of finite order, say, $n$.

It is well known that $K$ is separable over $K_\sigma = \{k \in K \mid \sigma(k) = k\}$ and $[K : K_\sigma] = n$ (see [A, Theorems 14 and 15]). Furthermore, $D_\sigma = \{d \in D \mid \sigma(d) = d\}$ is also Dedekind domain by [G, (36.1) and (37.2)] and $D$ is a finitely generated $D_\sigma$-module by [ZS, Corollary 1, p.265]. Since the center $\mathbb{Z}(R)$ of $R$ is $D_\sigma[x^n]$, it follows that $R$ is a finitely generated $C$-module, where $C = D_\sigma[x^n]$. Thus $R$ is a classical $C$-order in $K(x; \sigma)$ and so $R$ is a prime PI ring with $\text{dim}(R) = \text{dim}(R) = 2$ (see [MR, (6.4.8) and (6.5.4.)]), where $K(R)$ is the Krull dimension of $R$ and $\text{dim}(R)$ is the classical Krull dimension of $R$.

The following lemma is due to [Ro, (1.6.27)].

**Lemma 2.3.** Let $\sigma$ be an automorphism of $K$ with order $n$. Then

1. there is a one-to-one correspondence between $\text{Spec}(K[x; \sigma])$ and $\text{Spec}(K_\sigma[x^n])$, which is given by $P' \mapsto P' = P' \cap K_\sigma[x^n]$, where $P' \in \text{Spec}(K[x; \sigma])$.

2. If $P' = xK[x; \sigma]$, then $P' = x^nK_\sigma[x^n]$ and $P'K[x; \sigma] = P''$. If $P' \neq xK[x; \sigma]$, then $P' = f(x^n)K_\sigma[x^n]$ for some irreducible polynomial $f(x^n)$ in $K_\sigma[x^n]$ different from $x^n$ and $P'K[x; \sigma] = P''$.

**Lemma 2.4.** Let $\sigma$ be an automorphism of $D$ with order $n$. Then

1. There is a one-to-one correspondence between $\text{Spec}_0(R)$ and $\text{Spec}_0(C)$, which is given by $P \mapsto p = P \cap C$, where $P \in \text{Spec}_0(R)$.

2. If $P = xR$, then $P^n = pR$, where $p = P \cap C$. If $P \neq xR$, then $P = pR$, where $p = P \cap C$.

**Proof.** (1) Let $P \in \text{Spec}_0(R)$. Then it is clear that $p = P \cap C \in \text{Spec}_0(C)$. Conversely, let $p \in \text{Spec}_0(C)$. If $p \neq x^nC$, then $P = pK[x; \sigma] \cap R \in \text{Spec}_0(R)$ by Lemma 2.3 and [GW, (9.22)], and so $p \subseteq p_1 = P \cap C \in \text{Spec}_0(C)$. Hence $p = p_1$ by Proposition 2.1. If $p = x^nC$, then $P = xR \in \text{Spec}_0(R)$ with $p = P \cap C$. Hence the correspondence is onto.

To prove the correspondence is one to one, let $P$ and $P_1 \in \text{Spec}_0(R)$ with $P \cap C = p = P_1 \cap C$. We may assume that $P \neq xR$ and $P_1 \neq xR$. Then $PK[x; \sigma]$ and $P_1K[x; \sigma]$ both contain $pK[x; \sigma] \in \text{Spec}(K[x; \sigma])$ and so $PK[x; \sigma] = P_1K[x; \sigma]$ follows. Hence $P = PK[x; \sigma] \cap R = P_1$.

(2) $P \in \text{Spec}_0(R)$ with $p = P \cap C$. If $P = xR$ then $P^n = pR$ where $p = x^nC$. Suppose that $P \neq xR$. Let $P_1$ be an invertible prime ideal containing $pR$. By Proposition 2.1, $P_1$ is a minimal prime ideal of $R$. So either $P_1 = p_1[x; \sigma]$, where $p_1$ is a $\sigma$-prime ideal of $D$ or $P_1 \in \text{Spec}_0(R)$ by Proposition 2.1. If $P_1 = p_1[x; \sigma]$, then $P_1 \cap C = (p_1)_\sigma[x^n]$, a minimal prime ideal of $C[x^n]$, where $(p_1)_\sigma = p_1 \cap D_\sigma$, containing
\( \mathfrak{p} \) so that \( \mathfrak{p} = (\mathfrak{p}_1)_\sigma[x^n] \), a contradiction, because \( P \in \text{Spec}_0(R) \). Hence \( P_1 \in \text{Spec}_0(R) \). It follows that \( \mathfrak{p}_1 = P_1 \cap C \geq \mathfrak{p} \) and so \( \mathfrak{p}_1 = \mathfrak{p} \). Hence \( P = P_1 \) by (1). Since the invertible ideal \( \mathfrak{p}R \) is a finite product of invertible prime ideals (see [CS, Theorem 1.6 and Proposition 2.3]), we have \( \mathfrak{p}R = P^e \) for some \( e \geq 1 \). Then \( \mathfrak{p}K[x;\sigma] = P^e K[x;\sigma] = P^e \) implies \( e = 1 \). Hence \( P = \mathfrak{p}R \) follows.

**Lemma 2.5.** Let \( P \in \text{Spec}_0(R) \) with \( P \neq xR \). Then \( P_n \) is principal generated by a central polynomial in \( C_n \) for any \( n \in \text{Spec}(D_\sigma) \).

**Proof.** Let \( \mathfrak{p} = P \cap C \). Then \( \mathfrak{p}_n \) is principal by [M2, (3.1)], because \( C_n = (D_\sigma)_n[x^n] \) and \( (D_\sigma)_n \) is a discrete rank one valuation ring. Hence \( P_n \) is principal generated by a central element in \( C_n \) by Lemma 2.4.

**Lemma 2.6.** Let \( P \in \text{Spec}_0(R) \) with \( P \neq xR \). Then the following are equivalent:

1. \( P \nsubseteq M^2 \) for any maximal ideal \( M \) of \( R \).
2. \( P_n \nsubseteq (M_n)^2 \) for any \( n \in \text{Spec}(D_\sigma) \) and for any maximal ideal \( M \) of \( R \) with \( M \cap (D_\sigma \setminus n) = \emptyset \).

**Proof.** (1) \( \Rightarrow \) (2): Suppose that there is an \( n \in \text{Spec}(D_\sigma) \) and a maximal ideal \( M \) of \( R \) with \( M \cap (D_\sigma \setminus n) = \emptyset \) satisfying \( P_n \subseteq (M_n)^2 \). Then there is a \( c \in D_\sigma \setminus n \) with \( cP \subseteq M^2 \subseteq M \), which implies \( P \subseteq M \) and \( cR + M = R \). Hence \( P = (cR + M)P \subseteq M^2 \), a contradiction. Hence, for any \( n \in \text{Spec}(D_\sigma) \) and any maximal ideal \( M \) of \( R \) with \( M \cap (D_\sigma \setminus n) = \emptyset \), \( P_n \nsubseteq (M_n)^2 \).

(2) \( \Rightarrow \) (1): Suppose that there is a maximal ideal \( M \) of \( R \) with \( P \subseteq M^2 \). Then \( M \cap D \neq (0) \) by Proposition 2.1 and so \( n = M \cap D_\sigma \neq (0) \), which is a prime ideal of \( D_\sigma \) with \( M \cap (D_\sigma \setminus n) = \emptyset \). By the assumption, \( P_n \nsubseteq (M_\sigma)^2_n = M_\sigma^2_n \), a contradiction. Hence \( P \nsubseteq M^2 \) for any maximal ideal \( M \) of \( R \).

**Lemma 2.7.** Let \( P \in \text{Spec}_0(R) \) with \( P \neq xR \) and \( \mathfrak{p} = P \cap C \). Then \( \mathbb{Z}(R/P) = (C/\mathfrak{p}) \).

**Proof.** Since \( \mathbb{Z}(R/P) = \mathbb{Z}(K[x;\sigma]/P') \cap (R/P) \), it suffices to prove that \( \mathbb{Z}(K[x;\sigma]/P') = (K_\sigma[x^n]/P') \), where \( P' = K_\sigma[x^n] \cap P' \). We set \( K[x;\sigma] = K[x;\sigma]/P' \). It is clear that \( \mathbb{Z}(K[x;\sigma]) \supseteq (K_\sigma[x^n]/P') \). To prove the converse inclusion, let \( f(x^n) \in K_\sigma[x^n] \) be a monic polynomial with \( P' = f(x^n)K[x;\sigma] \) and deg \( f(x^n) = nl \). Write

\[
f(x^n) = x^{nl} + a_{l-1}x^{n(l-1)} + \cdots + a_1x^n + a_0, \quad \text{where } a_i \in K_\sigma.
\]

Suppose that \( a_0 = 0 \). Then \( f(x^n) = h(x^n)x^n \), where \( h(x^n) = x^{n(l-1)} + \cdots + a_1 \), shows that \( P' \subseteq xK[x;\sigma] \) and so \( P' = xK[x;\sigma] \), a contradiction. So we may assume that \( a_0 \neq 0 \). Note that

\[
K[x;\sigma] \cong K \oplus K\overline{x} \oplus \cdots \oplus K\overline{x}^{nl-1},
\]
as a ring and that

\[
\overline{x}^{nl} = -(a_{l-1}\overline{x}^{n(l-1)} + \cdots + a_1\overline{x}^n + a_0).
\]

Let \( g(x) = b_{nl-1}\overline{x}^{nl-1} + \cdots + b_1\overline{x} + b_0 \) be any element in \( \mathbb{Z}(K[x;\sigma]) \), where \( b_i \in K \). Then, for any \( k \in K \), \( kg(x) = g(x)k \) implies \( b_i\sigma^i(k) = b_ik \) for any \( i, 0 \leq i \leq nl - 1 \).
Suppose that there is an $i$ with $b_i \neq 0$ and $i = nj + s$ $(1 \leq s < n)$. Then $b_i \sigma^s(k) = b_k$ and so $\sigma^s(k) = k$ for all $k \in K$, a contradiction. Thus if $b_i \neq 0$, then $i = nj$, $0 \leq j \leq l - 1$. Next

\[ \overline{g(x)} = b_0 \overline{x} + b_1 \overline{x}^2 + \cdots + b_{n-2} \overline{x}^{n-1} + b_{n-1}(-a_{l-1} \overline{x}^{n(l-1)} - \cdots - a_1 \overline{x} - a_0) \quad \text{and} \]

\[ \overline{\sigma(x)} = \sigma(b_0) \overline{x} + \sigma(b_1) \overline{x}^2 + \cdots + \sigma(b_{n-2}) \overline{x}^{n-1} + \sigma(b_{n-1})(-a_{l-1} \overline{x}^{n(l-1)} - \cdots - a_1 \overline{x} - a_0). \]

Since $\overline{g(x)} = \overline{\sigma(x)}$, comparing the coefficients, we have $\sigma(b_{n-1}) = b_{n-1}$, that is, $b_{n-1} \in K_\sigma$ and so $\sigma(b_i) = b_i$ for all $0 \leq i \leq n-2$. Thus we have

\[ \overline{g(x)} = b_0 + b_1 \overline{x} + \cdots + b_{n-1} \overline{x}^{n-1} \quad \text{and} \quad \overline{b_1} \in K_\sigma. \]

Hence $\overline{g(x)} \in \langle K_\sigma[x^n]/p' \rangle$.

Let $P \in \text{Spec}_0(R)$ with $P \neq xR$. Since $\mathbb{Z}(R/P) = (C/p) \supset D_\sigma$ naturally, it follows from [R, (3.24)] that $R/P$ is a hereditary prime ring if and only if $(R/P)_n(\cong R_n/P_n)$ is a hereditary prime ring for any $n \in \text{Spec}(D_\sigma)$.

Let $m$ be any maximal ideal of $C$ with $m \supset p$. By lying over and going up theorems (see [MR, (10.2.9) and (10.2.10)]), there is a maximal ideal $M$ of $R$ with $M \cap C = m$ and $M \supset P$. Set $J = \bigcap \{M \mid M$ is a maximal ideal of $R$ with $m = M \cap C\}$. Since $\dim(R/J) = K(R/J) < K(R) = 2$, $M/J$ is a minimal prime ideal of $R/J$ and $J$ is a finite intersection of those $M$'s, that is, $J = M_1 \cap \ldots \cap M_k$ (see [CH, Lemma 1.16]). Thus we have the following lemma.

**Lemma 2.8.** With the notation above, the following hold:

1. $P \notin M_i^2$ if and only if $P_m \notin M_i^2$.
2. $M_i \supset M_i^2$ for any $i$ $(1 \leq i \leq k)$.
3. $\text{gl.dim} \ R_m = 2$ and $J(R_m) = M_1 m \cap \ldots \cap M_k m$.

**Proof.** (1) This is proved in the same way as in [MLP, Lemma 2].

(2) Set $M = M_i$ and $m_0 = M \cap D = (0)$, because $M \supset P$. If $x \in M$, then $M = m_0 + xR$ and $m_0$ is a maximal ideal of $D$ with $m_0 \supset m_0^2$. Thus $M^2 \subseteq m_0^2 + xR \subset m_0 + xR = M$. If $x \notin M$, then $m_0$ is a $\sigma$-prime ideal and $D/m_0$ is a semi-simple Artinian ring. Since $M \ni m_0[x; \sigma]$, we have

\[ \overline{M} = (M/m_0[x; \sigma]) \subset \overline{R} = (R/m_0[x; \sigma]) \cong (D/m_0)[x; \sigma], \]

which is hereditary by [MR, (7.5.3)]. Since $\overline{x} \notin \overline{M}$, $\overline{M}$ is principal by [CFH, Lemma 2.6]. So $(\overline{M})^2 \subset \overline{M}$ and thus $M^2 \subset M$ follows.

(3) It follows that $2 = \text{gl.dim} R \geq \text{gl.dim} R_m$. If $\text{gl.dim} R_m \leq 1$, then $R_m$ is hereditary, which is implies $M_m = P_m$. Hence $M = M_m \cap R = P_m \cap R = P$, a contradiction. Hence $\text{gl.dim} R_m = 2$. Since $R_m$ is a PI ring with the maximal ideals $M_1 m, \ldots, M_k m$, it is clear that $J(R_m) = M_1 m \cap \cdots \cap M_k m$.

**Proposition 2.9.** Let $\sigma$ be an automorphism of $D$ with order $n$ and let $P \in \text{Spec}_0(R)$ with $P \neq xR$. Then $\overline{R} = R/P$ is a hereditary prime ring if and only if $P \notin M^2$ for
any maximal ideal \( M \) of \( R \).

Proof. First note that \( \mathbb{Z}(\overline{R}) = \overline{C} = (C/p) \) by Lemma 2.7, where \( p = P \cap C \). Suppose that \( \overline{R} \) is a hereditary prime ring. Then \( \overline{C} \) is a Dedekind domain (see [MR, (13.9.16)]). Let \( M \) be a maximal ideal of \( R \). If \( P \nsubseteq M \), then \( P \nsubseteq M^2 \). So we may assume that \( P \subseteq M \). In order to prove \( P \nsubseteq M^2 \), we may assume that \( P \) is principal generated by a central element by Lemmas 2.5 and 2.6 and let \( m = M \cap C \), a maximal ideal of \( C \) properly containing \( p \). Then there are a finite number of maximal ideals \( M_1, \ldots, M_k \) of \( R \) lying over \( m \) such that \( J(\overline{R}_m) = (\overline{M}_1)_m \cap \cdots \cap (\overline{M}_k)_m \) and \( \overline{C}_m \) is a discrete rank one valuation ring, where \( M = M_1, \overline{M}_i = M_i/P \) and \( \overline{m} = (m/p) \). If \( k = 1 \), then \( \overline{R}_m \) is a local Dedekind prime ring so that it is a principal ideal ring. So \( \overline{M}_m = a\overline{R}_m \) for some \( a \in M_1 \) and \( M_m = aR_m + P_m \). Suppose that \( P \subseteq M^2 \). Then \( M_m = aR_m + P_m \subseteq aR_m + M_m J(R_m) \subseteq M_m \). Hence \( M_m = aR_m \) by Nakayama’s lemma, which is invertible. It follows from [HL, Proposition 1.3] that \( R_m \) is a principal ideal ring. So \( \text{gl.dim} R_m \leq 1 \), which contradicts Lemma 2.8. Hence \( P \nsubseteq M^2 \).

If \( k \geq 2 \), then \( \overline{M}_{1m}, \ldots, \overline{M}_{km} \) is a cycle by Lemma 1.1, because \( \overline{C}_m \) is a discrete rank one valuation ring. Suppose that \( P \subseteq M^2 \). Then \( \overline{M}_m = \overline{M}_m^2 \) implies

\[
M_m = (M_m^2) + P_m = (M_m)^2 = M_m^2.
\]

Let \( m_i \) be another maximal ideal of \( C \). Then \( M_{m_i} = R_{m_i} \) and so \( R_{m_i} = (M_{m_i})^2 = (M^2)_{m_i} \). Hence \( M = m \cap M_{m_i} = \cap (M^2)_{m_i} = M^2 \), which contradicts Lemma 2.8, where \( m_j \) runs over all maximal ideals of \( C \). Hence \( P \nsubseteq M^2 \).

Conversely, suppose that \( P \nsubseteq M^2 \) for any maximal ideal \( M \) of \( R \). Let \( m \) be a maximal ideals of \( C \) with \( m \supseteq p \) and \( n = m \cap D_\sigma \), a maximal ideal of \( D_\sigma \). Since \( (R_m)_m = R_m \) and \( (P_n)_m = P_m \), we may suppose that \( P \) is principal by Lemmas 2.5 and 2.6. It follows from Lemma 2.8 and [MLP, Lemma 3] that \( \overline{R}_m = R_m/P_m \) is a hereditary prime ring. Hence \( \overline{R} \) is a hereditary prime ring by [R, (3.24)].

Summarizing Propositions 2.1, 2.2, and 2.9, we have the following theorem:

**Theorem 2.10.** Let \( R = D[x; \sigma] \) be a skew polynomial ring over a commutative Dedekind domain, where \( \sigma \) is an automorphism of \( D \) and \( P \) be a prime ideal of \( R \). Then

1. \( P \) is a minimal prime ideal of \( R \) if and only if either \( P = p[x; \sigma] \), where \( p \) is either a non-zero \( \sigma \)-prime ideal of \( D \) or \( P \in \text{Spec}_0(R) \) with \( P \neq (0) \).

2. If \( P = p[x; \sigma] \), where \( p \) is a non-zero \( \sigma \)-prime ideal of \( D \), then \( R/P \) is a hereditary prime ring. In particular, \( R/P \) is a Dedekind prime ring if and only if \( p \in \text{Spec}(D) \).

3. If \( P \in \text{Spec}_0(R) \) with \( P = xR \), then \( R/P \) is a Dedekind prime ring. In particular, if the order of \( \sigma \) is infinite, then \( P = xR \) is the only minimal prime ideal belonging to \( \text{Spec}_0(R) \).

4. If \( P \in \text{Spec}_0(R) \) with \( P \neq xR \) and \( P \neq (0) \), then \( R/P \) is a hereditary prime ring if and only if \( P \nsubseteq M^2 \) for any maximal ideal \( M \) of \( R \).
3 Examples

Let $D = \mathbb{Z} \oplus \mathbb{Z}i$ be the Gauss integers, where $i^2 = -1$, and let $\sigma$ be the automorphism of $D$ with $\sigma(a + bi) = a - bi$, where $a, b \in \mathbb{Z}$, the ring of integers.

In this section, we will give some examples of minimal prime ideals of a skew polynomial ring over $D$, in order to display some of the various phenomena in section 2.

Let $p$ be a prime number. Then the following properties are well known in the elementary number theory:

1. If $p = 2$, then $2D = (1 + i)2D$ and $(1 + i)D$ is a prime ideal.
2. If $p = 4n + 1$, then $pD = \pi\sigma(\pi)D$ for some prime element $\pi$ with $\pi D + \sigma(\pi)D = D$.
3. If $p = 4n + 3$, then $pD$ is a prime ideal of $R$.

We let $R = D[x; \sigma]$ be the skew polynomial ring, $P = (x^2 + p)R \in \text{Spec}_0(R)$ and $\overline{R} = R/P$.

**Lemma 3.1.** If $p = 2$, then $\overline{R}$ is not a hereditary prime ring.

**Proof.** Let $M = (1 + i)D + xR$ be a maximal ideal of $R$. Then $M^2 = 2D \oplus (1 + i)Dx \oplus x^2R$ and so $M^2 \ni x^2 + 2$. Hence $\overline{R}$ is not a hereditary prime ring by Theorem 2.10.

In what follows, we suppose that $p \neq 2$ unless otherwise stated. Let $M$ be maximal ideal containing $x^2 + p$. First we will study in the case where $M \ni x$. Then $M = \pi D + xR$ for some prime element $\pi$ of $D$ with either $pD = \pi\sigma(\pi)D$ and $\pi D + \sigma(\pi)D = D$ if $p = 4n + 1$ or $pD = \pi D$ if $p = 4n + 3$.

**Lemma 3.2.** Let $M = \pi D + xR$ be a maximal ideal of $R$ with $M \ni P$. Then

1. If $p = 4n + 1$, then $M^2 \not\ni x^2 + p$ and $M = M^2 + P$, that is, $\overline{M}$ is idempotent.
2. If $p = 4n + 3$, then $M^2 \not\ni x^2 + p$ and $M \ni M^2 + P$, that is, $\overline{M}$ is not idempotent.

**Proof.** (1) It follows that $M^2 = \pi^2 D + xR$, because $D = \pi D + \sigma(\pi)D$. Suppose that $x^2 + p \in M^2$. Then $p \in \pi^2 D$ and so $\sigma(\pi)D = \pi D$ follows, a contradiction. Hence $M^2 \not\ni x^2 + p$. Since $\pi D = M \cap D \supseteq (M^2 + P) \cap D \supseteq M^2 \cap D = \pi^2 D$, we have either $(M^2 + P) \cap D = \pi D$ or $(M^2 + P) \cap D = \pi^2 D$. If $(M^2 + P) \cap D = \pi^2 D$, then $M^2 + P \ni \pi^2 + x^2 - (x^2 + p) = \pi^2 - p$, which implies $p \in \pi^2 D$, a contradiction as the above. So $(M^2 + P) \cap D = \pi D$ and thus $M^2 + P \ni \pi D + xR = M$. Hence $M = M^2 + P$ follows.

(2) It is easy to see that $M^2 \not\ni x^2 + p$ since $M^2 = p^2 D + pxR + x^2 R$. Suppose that $M = M^2 + P$. Then $x \in M^2 + P$ and write $x = p^2 d + px f(x) + x^2 g(x) + (x^2 + p) h(x)$, where $d \in D$, $f(x) = \sum f_i x^i$, $g(x) = \sum g_i x^i$ and $h(x) = \sum h_i x^i$, where $f_i, g_i, h_i \in D$. Then $1 = p\sigma(f_0) + ph_1$, a contradiction. Hence $M \ni M^2 + P$.  

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Next we will study a maximal ideal $M$ with $M \not
 \subseteq x$.

**Lemma 3.3.** Let $M$ be a maximal ideal of $R$ with $M \ni x^2 + p$ and $M \not
 \subseteq x$. Then
1. There is a prime number $q$ ($\not
 = p$) and a monic polynomial $f(x) \in M$ with $M = f(x)R + qR$.
2. If $\deg f(x) \geq 2$, then $M = P + qR$, $M^2 \not
 \subseteq x^2 + p$ and $\overline{M}$ is not idempotent.
3. If $\deg f(x) = 1$, then $q = 2$ and either $M = (x+1)R + 2R$ or $M = (x+i)R + 2R$.

**Proof.** (1) Since $M \cap D$ is a non-zero $\sigma$-prime ideal, there is a prime number $q$ with $M \cap D = qD$. Set $\tilde{R} = R/qD[x; \sigma] = \tilde{D}[x; \tilde{\sigma}]$, where $\tilde{D} = D/qD = (\mathbb{Z}/q\mathbb{Z}) \oplus (\mathbb{Z}/q\mathbb{Z})i$, a semi-simple Artinian ring. Since $\tilde{M} = M/qD[x; \sigma] \not
 \subseteq \tilde{x}$, it follows from [CFH, Lemma 2.6] that $\tilde{M} = f(x)\tilde{R}$ for some monic polynomial $f(x)$, where $f(x) \in \tilde{M}$. Hence we do not need to consider the maximal ideal $(x+i+1)R + 2R$. If $M = (x+1)R + 2R$, then it is easy to see that $M \not
 \subseteq x$, because $\tilde{M} = (x+\tilde{1})\tilde{R}$. Let $p = 2l + 1$ (note $p \not
 = 2$). Then $M \ni (x+1)^2 + 2(l-x) = x^2 + p$. Similarly we can prove that $(x+i)R + 2R \not
 \subseteq x$ and $(x+i)R + 2R \ni x^2 + p$.

From the proof of Lemma 3.3, we have

**Remark.** $M = (x+1)R + 2R$ and $N = (x+i)R + 2R$ are both maximal ideals of $R$ containing $x^2 + p$.

**Lemma 3.4.** If $p = 4n + 3$, then $\overline{R}$ is not a hereditary prime ring.

**Proof.** Let $M = (x+1)R + 2R$, a maximal ideal of $R$. Then $M^2 \ni (x+1)^2 + 4n + 1 = x^2 + p$. Hence $\overline{R}$ is not a hereditary prime ring by Theorem 2.10.

**Lemma 3.5.** If $p = 4n + 1$, then $\overline{R}$ is a hereditary prime ring, but not a Dedekind prime ring.

**Proof.** Let $M = (x+1)R + 2R$ and $N = (x+i)R + 2R$, the maximal ideals of
Then since \( M \)

First we will prove that \( N \)

for some \( f \)

\( \frac{R}{4R} \), and using the same notation in \( R \), we may suppose that

\[
x^2 + 1 = (x^2 + 2x + 1)f(x) + 2(x + 1)g(x)
\]

for some \( f(x) = f_n x^n + \cdots + f_1 x + f_0 \) and \( g(x) = g_n + x^{n+1} + \cdots + g_1 x + g_0 \), where \( f_i, g_j \in D \). Comparing the coefficients of \( x^j \) (\( 0 \leq j \leq n + 2 \)), we have

\[
1 = f_0 + 2g_0, \\
0 = 2\sigma(f_0) + f_1 + 2\sigma(g_0) + 2g_1, \\
1 = f_0 + 2\sigma(f_1) + f_2 + 2\sigma(g_1) + 2g_2, \\
0 = f_{j-2} + 2\sigma(f_{j-1}) + f_j + 2\sigma(g_{j-1}) + 2g_j (2 \leq j \leq n), \\
0 = f_{n-1} + 2\sigma(f_n) + 2\sigma(g_n) + 2g_{n+1}, \\
0 = f_n + 2\sigma(g_{n+1}).
\]

Here if \( \deg f(x) = 0 \), then \( f_1 = f_2 = g_2 = 0 \), and if \( \deg f(x) = 1 \), then \( f_2 = 0 \).

Adding the coefficients of \( x^{2j} \) and \( x^{2j+1} \), respectively, we have the following equations: Case 1, \( n \) is even number, say, \( n = 2l \).

\[
2 = 2\left( \sum_{j=0}^{l} f_{2j} + \sum_{j=1}^{l} \sigma(f_{2j-1}) \right) + 2\left( \sum_{j=0}^{l} g_{2j} + \sum_{j=1}^{l+1} \sigma(g_{2j-1}) \right) \tag{1}
\]

and

\[
0 = 2\left( \sum_{j=0}^{l} \sigma(f_{2j}) + \sum_{j=1}^{l} f_{2j-1} \right) + 2\left( \sum_{j=0}^{l} \sigma(g_{2j}) + \sum_{j=1}^{l+1} g_{2j-1} \right) \tag{2}
\]

Set \( \alpha = \sum_{j=0}^{l} f_{2j}, \ \beta = \sum_{j=1}^{l} f_{2j-1}, \ \gamma = \sum_{j=0}^{l} g_{2j}, \) and \( \delta = \sum_{j=1}^{l+1} g_{2j-1} \). Then adding (1) to (2), we have

\[
2 = 2(\alpha + \sigma(\alpha) + \beta + \sigma(\beta) + \gamma + \sigma(\gamma) + \delta + \sigma(\delta)) = 4c \text{ for some } c \in \mathbb{Z}, \text{ a contradiction. Hence } M^2 \not\equiv x^2 + p.
\]

Case 2, \( n = 2l + 1 \),

\[
2 = 2\left( \sum_{j=0}^{l} f_{2j} + \sum_{j=1}^{l+1} \sigma(f_{2j-1}) \right) + 2\left( \sum_{j=0}^{l+1} g_{2j} + \sum_{j=1}^{l+1} \sigma(g_{2j-1}) \right) \tag{3}
\]

and

\[
0 = 2\left( \sum_{j=0}^{l} \sigma(f_{2j}) + \sum_{j=1}^{l+1} f_{2j-1} \right) + 2\left( \sum_{j=0}^{l+1} \sigma(g_{2j}) + \sum_{j=1}^{l+1} g_{2j-1} \right). \tag{4}
\]

Adding (3) to (4), we have \( 2 = 4d \) for some \( d \in \mathbb{Z} \), a contradiction. Hence \( M^2 \not\equiv x^2 + p \).

Next suppose that \( N^2 \not\equiv x^2 + p \). Since \( N^2 = (x^2 - 1)R + 2(x + i)R + 4R \), as before, we may suppose that

\[
x^2 + 1 = (x^2 - 1)h(x) + 2(x + i)k(x)
\]

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for some $h(x) = h_n x^n + \cdots + h_1 x + h_0$ and $k(x) = k_{n+1} x^{n+1} + \cdots + k_1 x + k_0$, where $h_i, k_j \in D$. Comparing the coefficients of $x^j$ $(0 \leq j \leq n + 2)$, we have

$$
1 = -h_0 + 2k_0i,
0 = -h_1 + 2\sigma(k_0) + 2k_1i,
1 = (h_0 - h_2) + 2\sigma(k_1) + 2k_2i,
0 = h_{j-2} - h_j + 2\sigma(k_{j-1}) + 2k_2i \quad (3 \leq j \leq n),
0 = h_{n-1} + 2\sigma(k_n) + 2k_{n+1}i,
0 = h_n + 2\sigma(k_{n+1}).
$$

Here if $n = 0$, then $h_1 = h_2 = k_2 = 0$ and if $n = 1$, then $h_1 = h_2 = h_3 = k_3 = 0$. Adding the coefficients of $x^{2j}$ and $x^{2j+1}$, respectively, we have the following equations:

Case 1, $n = 2l$,

$$
2 = 2i\left(\sum_{j=0}^l k_{2j}\right) + 2\left(\sum_{j=0}^l \sigma(k_{2j+1})\right) \quad (5)
$$

$$
0 = 2\left(\sum_{j=0}^l \sigma(k_{2j})\right) + 2i\left(\sum_{j=0}^l k_{2j+1}\right) \quad (6)
$$

Operating $\sigma$ to (6) and multiplying it by $i$,

$$
0 = 2i\left(\sum_{j=0}^l k_{2j}\right) + 2\left(\sum_{j=0}^l \sigma(k_{2j+1})\right) \quad (7)
$$

Adding (5) to (7), we have $2 = 4i\left(\sum_{j=0}^l k_{2j}\right) + 4\sigma\left(\sum_{j=0}^l k_{2j+1}\right)$, a contradiction.

Case 2, $n = 2l + 1$,

$$
2 = 2i\left(\sum_{j=0}^{l+1} k_{2j}\right) + 2\left(\sum_{j=0}^l \sigma(k_{2j+1})\right) \quad (8)
$$

$$
0 = 2\left(\sum_{j=0}^{l+1} \sigma(k_{2j})\right) + 2i\left(\sum_{j=0}^l k_{2j+1}\right) \quad (9)
$$

Thus, by the same way as in the case $n = 2l$, $2 = 4i\left(\sum_{j=0}^{l+1} k_{2j}\right) + 4\sigma\left(\sum_{j=0}^l k_{2j+1}\right)$, a contradiction. Hence $N^2 \not\subseteq x^2 + p$, which complete the proof.

**Lemma 3.6.** Let $S = \{2^i | i = 0, 1, 2, \cdots\}$ be the central multiplicative set in $R$ and let $M$ be a maximal ideal of $R$ with $M \cap S = \emptyset$ and $M \supset P$. Then

1. $M^2 \supset P$ if and only if $M^2_S \supset P_S$.
2. $M^2 + P = M$ if and only if $(M^2 + P)_S = M_S$.

**Proof.** (1) If $M^2 \supset P$, then it is clear that $(M^2)_S \supset P_S$. Conversely suppose $M^2_S \supset P_S$. Then there is an $s \in S$ with $sP \subseteq M^2$. Since $sR + M = R$, we have $P = (sR + M)P \subseteq M^2$. 

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(2) This is proved in the same way as in (1).

Summarizing Lemmas 3.1 \(\sim\) 3.6, we have

**Proposition 3.7.** Let \( p \) be a prime number and \( P = (x^2 + p)R \). Then

1. If \( p = 2 \), then \( \overline{R} \) is not a hereditary prime ring.
2. If \( p = 4n + 3 \), then \( \overline{R} \) is not a hereditary prime ring and \( \overline{R}_S = R_S/P_S \) is a Dedekind prime ring, where \( S = \{2^i| i = 0, 1, 2, \cdots \} \).
3. If \( p = 4n + 1 \), then \( \overline{R} \) is a hereditary prime ring but not a Dedekind prime ring.

**Proof.** (1) This follows from Lemma 3.1

(2) By Lemma 3.4, \( \overline{R} \) is not a hereditary prime ring. Let \( M \) be a maximal ideal of \( R \) with \( M \supset P \) and \( M \cap S = \emptyset \). Then, by Lemmas 3.2, 3.3 and 3.6, \( (M^2)_S \nsubseteq P_S \) and \( \overline{M}_S \supset \overline{M^2}_S \). Hence \( \overline{R}_S \) is a Dedekind prime ring by [MR, (5.6.3)].

(3) \( \overline{R} \) is a hereditary prime ring but not Dedekind by Lemma 3.5.

We will end the paper with two remarks.

1. Let \( P = p[x; \sigma] \) be a minimal prime ideal of \( R \), where \( p \) is a non-zero \( \sigma \)-prime ideal of \( D \). Then there is a prime number \( p \) with \( p = pD \). If \( p = 4n + 1 \), then \( \overline{R} = R/P \) is a hereditary prime ring but not Dedekind. If \( p = 4n + 3 \), then \( \overline{R} = R/P \) is a Dedekind prime ring.

2. Let \( P' = (x^2 + \frac{1}{2})K[x; \sigma] \in \text{Spec}_0(K[x; \sigma]) \), where \( K = \mathbb{Q} \oplus \mathbb{Q}i \) and \( \mathbb{Q} \) is the field of rational numbers. Then \( P = P' \cap R = (2x^2 + 1)R \in \text{Spec}_0(R) \) and \( 2x^2 + 1 \) is not a monic polynomial (as it has been mentioned in the introduction, Hillman only considered monic polynomials).

**References**


