Note

The Ramsey numbers for disjoint unions of trees

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Received 1 March 2006; received in revised form 30 May 2006; accepted 25 June 2006

Abstract

For given graphs G and H, the Ramsey number R(G, H) is the smallest natural number n such that for every graph F of order n: either F contains G or the complement of F contains H. In this paper, we investigate the Ramsey number R(∪G, H), where G is a tree and H is a wheel Wm or a complete graph Km. We show that if n ≥ 3, then R(kSn, Wm) = (k + 1)n for k ≥ 2, even n and R(kSn, Wm) = (k + 1)n − 1 for k ≥ 1 and odd n. We also show that R(∪n=1i=1Tni, Km) = R(Tnk, Km) + ∑i=1n−1ni.

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Keywords: Ramsey number; Star; Wheel; Tree

1. Introduction

For given graphs G and H, the Ramsey number R(G, H) is defined as the smallest positive integer n such that for any graph F of order n, either F contains G or the complement of F contains H, where F is the complement of F.

In 1972, Chvátal and Harary [6] established a useful lower bound for finding the exact Ramsey numbers R(G, H), namely R(G, H) ≥ (χ(G) − 1)(c(H) − 1) + 1, where χ(G) is the chromatic number of G and c(H) is the number of vertices of the largest component of H. Since then the Ramsey numbers R(G, H) for many combinations of graphs G and H have been extensively studied by various authors, see a nice survey paper [9].

Let Pn be a path with n vertices and let Wm be a wheel of m + 1 vertices that consists of a cycle Cm with one additional vertex being adjacent to all vertices of Cm. A star Sn is the graph on n vertices with one vertex of degree n − 1, called the center, and n − 1 vertices of degree 1. Tn is a tree with n vertices and a cocktail-party graph Hs is the graph which is obtained by removing s disjoint edges from K2s.

Several results on Ramsey numbers have been obtained for wheels. For instance, Baskoro et al. [1] showed that for even m ≥ 4 and n ≥ (m/2)(m − 2), R(Pn, Wm) = 2n − 1. They also showed that R(Pn, Wm) = 3n − 2 for odd m ≥ 5, and n ≥ ((m − 1)/2)(m − 3).

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0012-365X/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.disc.2006.06.011
For a combination of stars with wheels, Surahmat et al. [10] investigated the Ramsey numbers for large stars versus small wheels. Their result is as follows.

**Theorem 6.** For any $m \geq 1$, 

\[
R(S_n, W_m) = \begin{cases} 
2n + 1 & \text{if } n \text{ is even,} \\
2n - 1 & \text{if } n \text{ is odd.}
\end{cases}
\]

For odd $m$, Chen et al. have shown in [4] that $R(S_n, W_m) = 3n - 2$ for $m \geq 5$ and $n \geq m - 1$. This result was strengthened by Hasmawati et al. in [8], by showing that this Ramsey number remains the same, as given in the following theorem.

**Theorem 2 (Hasmawati et al. [8]).** If $m$ is odd and $n \geq ((m + 1)/2) \geq 3$, then $R(S_n, W_m) = 3n - 2$.

If $n \leq (m + 2)/2$, Hasmawati [7] gave $R(S_n, W_m) = n + m - 2$ for even $m$ and odd $n$, or $R(S_n, W_m) = n + m - 1$, otherwise.

Let $G$ be a graph. The number of vertices in a maximum independent set of $G$ denoted by $\alpha(G)$, and the union of $s$ vertices-disjoint copies of $G$ denoted $sG$.

Burr et al. in [3], showed that if the graph $G$ has $n_1$ vertices and the graph $H$ has $n_2$ vertices, then

\[
n_1 s + n_2 t - D \leq R(sG, tH) \leq n_1 s + n_2 t - D + k,
\]

where $D = \min\{\alpha_0(G), t\alpha_0(H)\}$ and $k$ is a constant depending only on $G$ and $H$.

In the following theorem Chvátal gave the Ramsey number for a tree versus a complete graph.

**Theorem 3 (Chvátal [5]).** For any natural number $n$ and $m$, $R(T_n, K_m) = (n - 1)(m - 1) + 1$.

In this paper, we determine the Ramsey numbers $R(\cup G, H)$ of a disjoint union of a graph $G$ versus a graph $H$, where $G$ is either a star or a tree, and $H$ is either a wheel or a complete graph.

The results are presented in the next three theorems.

**Theorem 4.** If $m$ is odd and $n \geq (m + 1)/2 \geq 3$, then $R(kS_n, W_m) = 3n - 2 + (k - 1)n$.

**Theorem 5.** For $n \geq 3$,

\[
R(kS_n, W_m) = \begin{cases} 
(k + 1)n & \text{if } n \text{ is even and } k \geq 2, \\
(k + 1)n - 1 & \text{if } n \text{ is odd and } k \geq 1.
\end{cases}
\]

**Theorem 6.** Let $n_i \geq n_{i+1}$ for $i = 1, 2, \ldots, k - 1$. If $n_i \geq (n_i - n_{i+1})(m - 1)$ for any $i$, then $R(\cup_{i=1}^k T_{n_i}, K_m) = R(T_{n_k}, K_m) + \sum_{i=1}^{k-1} n_i$ for an arbitrary $m$.

Before proving the theorems, we present some notations used in this note. Let $G(V, E)$ be a graph. For any vertex $v \in V(G)$, the neighborhood $N(v)$ is the set of vertices adjacent to $v$ in $G$. Furthermore, we define $N[v] = N(v) \cup \{v\}$.

The degree of a vertex $v$ in $G$ is denoted by $d_G(v)$. The order of $G$, $|G|$ is the number of its vertices, and the minimum (maximum) degree of $G$ is denoted by $\delta(G)$ ($\Delta(G)$). For $S \subseteq V(G)$, $G[S]$ represents the subgraph induced by $S$ in $G$.

If $G$ is a graph and $H$ is a subgraph of $G$, then denote $V(G) \setminus V(H)$ by $G \setminus H$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The union $G = G_1 \cup G_2$ has the vertex set $V = V_1 \cup V_2$ and the edge set $E = E_1 \cup E_2$. Their join, denoted $G_1 + G_2$, is the graph with the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$.

2. The proofs of theorems

**Proof of Theorem 4.** Let $m$ be odd and $n \geq (m + 1)/2 \geq 3$. We shall use an induction on $k$. For $k = 1$, we have $R(S_n, W_m) = 3n - 2$ (by Theorem 2). Assume the theorem holds for any $r < k$, namely $R(rS_n, W_m) = (3n - 2) + (r - 1)n$.

We will show that $R(kS_n, W_m) = (3n - 2) + (k - 1)n$. If $m$ is odd, then $R(S_n, W_m) = 3n - 2$ (by Theorem 2). Assume the theorem holds for any $r < k$, namely $R(rS_n, W_m) = (3n - 2) + (r - 1)n$.
Let $F$ be a graph with $|F| = 3n - 2 + (k - 1)n$. Suppose that $\overline{F}$ contains no $W_m$. Since $|F| \geq R(rS_n, W_m)$, then $F \supseteq (k - 1)S_n$. Let $A = F \setminus (k - 1)S_n$ and $T = F[A]$. Thus, $|T| = 3n - 2$. Since $\overline{T}$ contains no $W_m$, then by Theorem 2, $T \supseteq S_n$. Thus, $F$ contains $kS_n$. Hence, we have $R(kS_n, W_m) \leq (3n - 2) + (k - 1)n$.

On the other hand, it is not difficult to see that $F_1 = K_{kn - 1} \cup 2K_{n-1}$ contains no $kS_n$ and its complement contains no $W_m$. Observe that $F_1$ has $3n - 3 + (k - 1)n$ vertices. Therefore, we have $R(kS_n, W_m) \geq (3n - 2) + (k - 1)n$, and the assertion follows. □

Proof of Theorem 5. Let $n$ be even, $n \geq 4$ and $k \geq 2$. Consider $F = (H_{(kn - 2)/2} + K_1) \cup H_{n/2}$. Clearly, graph $F$ has $(k + 1)n - 1$ vertices and contains no $kS_n$. Its complement contains no $W_4$. Hence, $R(kS_n, W_4) \geq (k + 1)n$. We will prove that $R(kS_n, W_4) = (k + 1)n$ for $k \geq 2$. First we will show that $R(2S_n, W_4) = 3n$.

Let $F_1$ be a graph of order $3n$. Suppose $\overline{F_1}$ contains no $W_4$. By Theorem 1, we have $F_1 \supseteq S_n$. Let $V(S_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ with center $v_0$, $A = F_1 \setminus S_n$ and $T = F_1[A]$. Thus $|T| = 2n$.

If there exists $u \in T$ with $d_T(u) \geq (n - 1)$, then $T$ contains $S_n$. Hence $F_1$ contains $2S_n$. Therefore we assume that for every vertex $u \in T$, $d_T(u) \leq (n - 2)$. Let $u, w \in T$ where $(u, w) \notin E(T)$. Consider $H = N[u] \cup N[w]$, $Q = T \setminus H$, $Z = N(u) \cap N(w)$, and $X = H \setminus \{u, w\}$ (see Fig. 1).

By contradiction suppose $d(u) \leq n - 3$. Then $0 \leq |Z| \leq n - 3, 2 \leq |H| \leq 2n - 3$ and $2n - 2 \geq |Q| \geq 3 + |Z|$. Observe that every $g \in Q$ is adjacent to at least $|Q| - 2$ other vertices of $Q$. (Otherwise, there exists $q \in Q$ which is not adjacent to at least two other vertices of $Q$, say $q_1$ and $q_2$. Then $\overline{F}$ will contain a $W_4 = \{q_1, u, q_2, w\}$ with $w$ as a hub, a contradiction.) Then, for all $q \in Q$, $d_Q(q) \geq |Q| - 2$.

Let $E(X \setminus Z, Q) = \{uv : u \in X \setminus Z, v \in Q\}$. If there exists $x \in X \setminus Z$ not adjacent to at least two vertices of $Q$, say $q_1$ and $q_2$, then $\overline{T}$ will contain $W_4 = \{q_1, x, q_2, u, w\}$ with $w$ or $u$ as a hub, a contradiction. Hence every $x \in X \setminus Z$ is adjacent to at least $|Q| - 1$ vertices in $Q$. Therefore, we have $|E(X \setminus Z, Q)| \geq |X \setminus Z| \cdot (|Q| - 1) - 1$.

On the other hand, every vertex $g \in Q$ is incident with at most $(n - 2) - d_Q(q) \leq (n - 2) - (|Q| - 2) = n - |Q|$ edges from $X \setminus Z$. Thus $|E(X \setminus Z, Q)| \leq |Q| \cdot (n - |Q|)$.

Now, we will show that $|X \setminus Z| \cdot (|Q| - 1) > |Q| \cdot (n - |Q|)$, which leads to a contradiction.

Writing $|X \setminus Z| \cdot (|Q| - 1) = |X \setminus Z| \cdot |Q| - |X \setminus Z|$ and substituting $|X \setminus Z| = 2n - 2 - |Q| - |Z|$, we obtain $|X \setminus Z| \cdot (|Q| - 1) = |Q| \cdot (n - |Q|) + |Q| \cdot (n - 2 - |Z|) + |Q| \cdot (n - (n - 2 - |Z|) - |Z| - n - (n - 2 - |Z|)) > 0$. Thus $|X \setminus Z| \cdot (|Q| - 1) > |Q| \cdot (n - |Q|)$. Hence there is no $u \in T$ such that $d(u) \leq n - 3$.

![Fig. 1. The illustration of the proof of $R(2S_n, W_4) \leq 3n$.](image-url)
Without loss of generality, assume \( F \) as \( S \). Suppose \( v \) \( k \) \( q \) and create a new star \( W \) which is impossible. Hence \( F \) is not a complete graph. Now, choose two vertices in \( Q \) which are not adjacent, call \( q_1 \) and \( q_2 \). Let \( Y = \{ q_1, q_2 \} \cup \{ u, w \} \), it is clear that \( Y \) is an independent set.

If there exists a vertex \( v \) in \( V(S_n) \) adjacent to at most one vertex in \( Y \) say \( q_1 \), then \( \{ v, u, q_1, q_2, w \} \) will induce a \( W_4 \) in \( F_1 \), with a hub \( w \), a contradiction. Therefore, every vertex \( v \in V(S_n) \) is adjacent to at least two vertices in \( Y \). Suppose \( v_0 \) and \( v_j \) in \( V(S_n) \) are adjacent to \( y_1 \) and \( y_2 \) in \( Y \) respectively. Note that at least two \( y_i \)'s are distinct.

Without loss of generality, assume \( y_1 \neq y_3 \). Since \( Y \) is independent, then we have two new stars, namely \( S_{n_1} \) and \( S_{n_2} \), where \( V(S_{n_1}) = S_n \setminus \{ v_0 \} \cup \{ y_1 \} \) with \( v_0 \) as the center and \( V(S_{n_2}) = N[y_3] \cup \{ v_j \} \) with \( y_3 \) as the center (see Fig. 1). So, we have \( F_1 \supseteq 2S_n \). Hence, \( R(2S_n, W_4) = 3n \).

Now, assume the theorem holds for every \( r < k \). We will show that \( R(kS_n, W_4) = (k + 1)n \). Let \( F_2 \) be a graph of order \((k + 1)n\), Suppose \( F_2 \) contains no \( W_4 \). We will show that \( F_2 \supseteq kS_n \). By induction, \( F_2 \supseteq (k - 1)S_n \). Denote the \( (k - 1)S_n \) as \( S_{n_1}^{s_1}, S_{n_2}^{s_2}, \ldots, S_{n_k}^{s_k} \) with the center \( v_1, v_2, \ldots, v_{k-1} \), respectively. Writing \( A' = F_2 \setminus (k - 1)S_n \) and \( T' = F_2[A'] \). Thus \( |T'| = 2n \).

Similarly, as in the case \( k = 2 \), every vertex \( u \in T' \), must have degree \( n - 2 \). Next, let \( u', w' \in T \) where \((u', w') \notin E(T') \), \( H' = N[u'] \cup N[w'] \), \( Q' = T' \setminus H' \), and \( Y' = \{ q_1, q_2 \} \cup \{ u', w' \} \), where \( q_1, q_2 \in Q' \) and \((q_1, q_2) \notin E(T') \) (see Fig. 2).

If the vertex \( v \in V((k - 1)S_n) \) is adjacent to at most one vertex in \( Y' \), say \( u' \), then \( F_2 \) will contain \( W_4 = \{ u', q_1, v, q_2, w' \} \) with \( w' \) as a hub, a contradiction.

Therefore, every vertex \( v \in V((k - 1)S_n) \) is adjacent to at least two vertices in \( Y' \). Suppose \( v_1 \) and \( s \in V(S_n) \) are adjacent to \( u', q_1 \) and to \( u', w' \), respectively (see Fig. 2). Then, we will alter \( s_n^k \) into \( s_n^l \) with \( V(S_n^l) = (S_n^{s_1}[s]) \cup \{ q_1 \} \) and create a new star \( S_n^k \) where \( V(S_n^k) = N[u'] \cup \{ s \} \) with the center \( u' \). Hence, we now have \( k \) disjoint stars, namely \( S_n^{s_1}, S_n^{s_2}, S_n^{s_3}, \ldots, S_n^{s_{k-1}} \) and \( S_n^k \). Therefore, we have \( R(kS_n, W_4) = (k + 1)n \).

Let \( n \) be odd. Consider \( F_3 = K_{kn-1} \cup K_{n-1} \). Clearly, the graph \( F_3 \) has order \((k + 1)n - 2 \), without containing \( kS_n \) and \( F_3 \) contains no \( W_4 \). Hence, \( R(kS_n, W_4) \geq (k + 1)n - 1 \). To obtain the Ramsey number we use an induction on \( k \). For \( k = 1 \), we have \( R(S_n, W_4) = 2n - 1 \). Suppose the theorem holds for every \( r < k \). We show that \( R(kS_n, W_4) = (k + 1)n - 1 \).

Let \( F_4 \) be a graph of order \((k + 1)n - 1 \). Suppose \( F_4 \) contains no \( W_4 \). By the assumption, \( F_4 \) contains \((k - 1)S_n \). Let
\[ B = F_4 \setminus (k - 1)S_n \] and \[ L = F_4[4] \]. Thus \(|L| = 2n - 1\). Since \( L \) contains no \( W_4 \), then by Theorem 1, \( L \supset S_n \). Therefore, \( F_4 \) contains \( kS_n \). The proof is now complete. \( \square \)

3 Proof of Theorem 6. Let \( n_i \geq n_{i+1} \) and \( n_i \geq (n_i - n_{i+1})(m - 1) \) for any \( i \). Since \( F = (m - 2)K_{n_i-1} \cup K_{\sum_{i=1}^k n_i - 1} \) has no \( \bigcup_{i=1}^k T_{n_i} \) and its complement contains no \( K_m \), then \( R(\bigcup_{i=1}^k T_{n_i}, K_m) \geq (m - 1)(n_k - 1) + \sum_{i=1}^{k-1} n_i + 1 \). We fix \( m \) and apply an induction on \( k \). For \( k = 2 \), we show that \( R(T_{n_1} \cup T_{n_2}, K_m) = (m - 1)(n_2 - 1) + n_1 + 1 \).

Let \( F_1 \) be a graph with \(|F_1| = (m - 1)(n_2 - 1) + 1 + n_1 \). Suppose \( \overline{F_1} \) contains no \( K_m \). Since \( n_1 \geq n_2 \), then we can write
\[ n_1 - n_2 = q \geq 0. \]
Substitute \( n_2 = n_1 - q \). Then we obtain
\[ |F_1| = (m - 1)(n_1 - q - 1) + n_1 + 1 = (m - 1)(n_1 - 1) - q(m - 1) + n_1 + 1 \]
or
\[ |F_1| = (m - 1)(n_1 - 1) + 1 + [n_1 - (n_1 - n_2)(m - 1)]. \]
Noting \( n_1 - (n_1 - n_2)(m - 1) \geq 0 \), it can be verified that
\[ |F_1| \geq (m - 1)(n_1 - 1) + 1 \text{ i.e. } |F_1| \geq R(T_{n_1}, K_m). \]
Hence, \( F_1 \supset T_{n_1} \). Now, let \( A = F_1 \setminus T_{n_1} \), and \( H = F_1[A] \). Then
\[ |H| = (m - 1)(n_2 - 1) + 1. \]
Since \( H \) contains no \( K_m \), then by Theorem 3, \( H \supset T_{n_2} \). Therefore, \( F_1 \) contains a subgraph \( T_{n_1} \cup T_{n_2} \).

Next, assume the theorem holds for all \( r < k \), namely \( R(\bigcup_{i=1}^{r'} T_{n_i}, K_m) \geq (m - 1)(n_{r'} - 1) + \sum_{i=1}^{r'} n_i + 1 \). We shall show that \( R(\bigcup_{i=1}^{k} T_{n_i}, K_m) \geq (m - 1)(n_k - 1) + \sum_{i=1}^{k-1} n_i + 1 \). Take an arbitrary graph \( F_2 \) with order \((m - 1)(n_k - 1) + \sum_{i=1}^{k-1} n_i + 1\). Suppose \( \overline{F_2} \) contains no \( K_m \). By induction, \( F_2 \) contains \( \bigcup_{i=1}^{k-1} T_{n_i} \).

Writing \( B = F_2 \setminus \bigcup_{i=1}^{k-1} T_{n_i} \), and \( Q = F_2[B] \). Then \(|Q| = (m - 1)(n_k - 1) + 1 \). Since \( \overline{Q} \) contains no \( K_m \), then \( Q \) contains \( T_{n_k} \). Hence \( F_2 \) contains \( \bigcup_{i=1}^{k} T_{n_i} \). Therefore, we have \( R(\bigcup_{i=1}^{k} T_{n_i}, K_m) = (m - 1)(n_k - 1) + \sum_{i=1}^{k-1} n_i + 1 \).

The proof is now complete. \( \square \)

3. Uncited reference

[2].

References