THE RAMSEY NUMBERS FOR DISJOINT UNION OF STARS

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Abstract. The Ramsey number for a graph $G$ versus a graph $H$, denoted by $R(G; H)$, is the smallest positive integer $n$ such that for any graph $F$ of order $n$, either $F$ contains $G$ as a subgraph or $\overline{F}$ contains $H$ as a subgraph.

In this paper, we investigate the Ramsey numbers for union of stars versus small cycle and small wheel. We show that if $n_i \geq 3$ for $i = 1, 2, \ldots, k$ and $n_i \geq n_{i+1} \geq \sqrt{n_i} - 2$, then $R(\bigcup_{i=1}^{k} S_{n_i}, C_4) = \sum_{i=1}^{k} n_i + k + 1$ for $k \geq 2$.

Furthermore, we show that if $n_i$ is odd and $2n_{i+1} \geq n_i$ for every $i$, then $R(\bigcup_{i=1}^{k} S_{n_i}, W_4) = R(S_{n_k}, W_4) + \sum_{i=1}^{k-1} n_i$ for $k \geq 1$.

Key words and Phrases: Ramsey number, Cycle, Wheel

1. Introduction

For given graphs $G$ and $H$, the Ramsey number $R(G, H)$ is defined as the smallest positive integer $n$ such that for any graph $F$ of order $n$, either $F$ contains $G$ or $\overline{F}$ contains $H$, where $\overline{F}$ is the complement of $F$. Chvátal and Harary [4]...
Let $G$ denote whose partite sets are of order complete bipartite $sG$. S. A. Burr et al. in [3], showed that if the graph spokes. $m$ have also been investigated. Let $S$. In particular, the Ramsey numbers for combinations involving union of stars and $H$ have been extensively studied by various authors, see a nice survey paper [8]. Since then the Ramsey numbers $R(G, H)$ for many combinations of graphs $G$ and $H$ have been extensively studied by various authors, see a nice survey paper [8]. In particular, the Ramsey numbers for combinations involving union of stars have also been investigated. Let $S_n$ be a star on $n$ vertices and $C_m$ be a cycle on $m$ vertices. We denote the complete bipartite whose partite sets are of order $n$ and $p$ by $K_{n,p}$.

For a combination of stars with wheels, Surahmat et al. [9] determined the Ramsey numbers for large stars versus small wheels. Their result is as follows.

**Theorem A.** (Surahmat and E. T. Baskoro, [9]) For $n \geq 3$, $R(S_n, W_4) = \begin{cases} 2n + 1, & \text{if } n \text{ is even,} \\ 2n - 1, & \text{if } n \text{ is odd.} \end{cases}$

Parsons in [7] considered about the Ramsey numbers for stars versus cycles as presented in Theorem A.

**Theorem B.** (Parsons’s upper bound, [7]) For $p \geq 2$, $R(S_{1+p}, C_4) \leq p + \sqrt{p} + 1$.

Hasanawati et al. in [6] and [5] proved that $R(S_6, C_4) = 8$, and $R(S_6, K_{2,m}) = 13$ for $m = 5$ or $6$ respectively.

Let $G$ be a graph. The number of vertices in a maximum independent set of $G$ denoted by $\alpha_0(G)$, and the union of $s$ vertices-disjoint copies of $G$ denoted by $sG$. S. A. Burr et al. in [3], showed that if the graph $G$ has $n_1$ vertices and the graph $H$ has $n_2$ vertices, then

$$n_1s + n_2t - D \leq R(sG, tH) \leq n_1s + n_2t - D + k,$$
Let there exists $n \in \mathbb{R}$ of a star versus a wheel. Their results are given in the next theorem.

**Theorem C.** [1] For $n \geq 3$, 
\[ R(kS_n, W_4) = \begin{cases} 
(k+1)n, & \text{if } n \text{ is even and } k \geq 2, \\
(k+1)n-1, & \text{if } n \text{ is odd and } k \geq 1. 
\end{cases} \]

2. Main Results

In this paper, we study the Ramsey numbers for disjoint union of stars versus small cycle and small wheel. The results are presented in the next two theorems. Before present these theorems let us present the lemma as follow.

**Lemma 2.1.** For $k \geq 2$ and $n_i \geq 3$ for every $i$, 
\[ R(\bigcup_{i=1}^{k} S_{n_i+1}, C_4) \geq \sum_{i=1}^{k} n_i + k + 1. \]

**Proof.** Let $n_i \geq 3$ for every $i$ and $k \geq 2$. Consider $F \cong K_{\sum_{i=1}^{k}(n_i+1)-1} \cup K_1$. Graph $F$ has $\sum_{i=1}^{k} n_i + k$ vertices, however it contains no $\bigcup_{i=1}^{k} S_{n_i+1}$. It is easy to see that $\overline{F}$ is isomorphic with $K_1, \sum_{i=1}^{k}(n_i+1)-1$. So, $\overline{F}$ contains no $C_4$. Hence, $R(\bigcup_{i=1}^{k} S_{n_i+1}, C_4) \geq \sum_{i=1}^{k}(n_i + 1) + 1$. \hfill \Box

**Theorem 2.2.** Let $n_i \geq 3$ for $i = 1, 2, \ldots, k$. If $n_i \geq n_{i+1} \geq \sqrt{n_i} - 2$, then $R(\bigcup_{i=1}^{k} S_{n_i+1}, C_4) = \sum_{i=1}^{k} n_i + k + 1$ for $k \geq 2$.

**Proof.** For $k = 2$, we show that $R(S_{n_1+2} \cup S_{n_2+2}, C_4) = n_1 + n_2 + 3$. Let $F_1$ be a graph of order $n_1 + n_2 + 3$ for $n_1, n_2 \geq 3$. Suppose $\overline{F_1}$ contains no $C_4$. Since $n_2 + 2 \geq \sqrt{n_1}$, then $|F_1| \geq n_1 + \sqrt{n_1} + 1$. By Parsons's upper bound, we have $|F_1| \geq R(S_{n_1+1}, C_4)$ for $n_1 \geq 3$. Thus $F_1 \cong S_{n_1+1}$. Let $V(S_{n_1+1}) = \{v_0, \ldots, v_{n_1}\}$ with center $v_0$. Write $T = F_1 \setminus S_{n_1+1}$. Thus $|T| = n_1 + 2$. If there exists $v \in T$ with $d_T(v) \geq n_2$, then $T$ contains $S_{n_2+2}$. Hence $F_1$ contains $S_{n_1+2} \cup S_{n_2+2}$. Therefore, we assume that for every vertex $v \in T$, $d_T(v) \leq (n_2 - 1)$.

Let $u$ be any vertex in $T$. Write $Q = T \setminus N_T[u]$. Clearly, $|Q| \geq 2$. Observe that if there exists $s \in F_1$ where $s \neq u$ which is not adjacent to at least two vertices in $Q$, then $F_1[\{s, u\} \cup Q]$ will contains $C_4$, a contradiction. Hence, for the remaining of the proof we will use the following assumption.

**Assumption 1.** Every vertex $s \in F_1$, $s \neq u$ is not adjacent to at most one vertex in $Q$.

Let $s$ be adjacent to at least $n_2 - |N_T(u)|$ vertices in $S_{n_1+1} - v_0$, call them $v_1, \ldots, v_{n_2 - |N_T(u)|}$. Observe that $n_2 - |N_T(u)| = |Q| - 1$. By Assumption 1, vertex
\(v_0\) is adjacent to at least \(|Q| - 1\) vertices in \(Q\), namely \(q_1, \ldots, q_{n_2 - |N_T(u)|}\). Then we have two new complete bipartite graphs, namely \(S'_1 + n_1\) and \(S_1 + n_2\), where

\[
V(S'_1 + n_1) = (S_1 + n_1 \setminus \{v_1, \ldots, v_{n_2 - |N_T(u)|}\}) \cup \{q_1, \ldots, q_{n_2 - |N_T(u)|}\}
\]

with \(v_0\) as the center and

\[
V(S_1 + n_2) = N_T[u] \cup \{v_1, \ldots, v_{n_2 - |N_T(u)|}\}
\]

with \(u\) as the center. Hence, we have \(F_1 \supseteq S_1 + n_2 \cup S_1 + n_2\).

Now, we assume that \(u\) is adjacent to at most \(n_2 - |N_T(u)| - 1\) vertices in \(S_1 + n_1 - v_0\). This means \(u\) is not adjacent to at least \(|N_T(u)| + 1\) vertices in \(S_1 + n_1 - v_0\). Let \(Y = \{y \in S_1 + n_1 - v_0 : yu \notin E(F_1)\}\). Then \(|Y| \geq |N_T(u)| + 1 \geq 1\).

Suppose for every \(y \in Y\), there exists \(r \in N_T(u)\) such that \(yr \notin E(F_1)\). Since \(|N_T(u)| < |Y|\), then there exists \(r_0 \in N_T(u)\) so that \(r_0\) is not adjacent to at least two vertices in \(Y\), say \(y_1\) and \(y_2\). This implies, \(F_1[u, r_0, y_1, y_2]\) forms a \(C_4\), a contradiction. Hence, there exists \(y' \in Y\) so that \(y'\) is adjacent to all vertices in \(N_T(u)\). Furthermore, by Assumption 1 we have that \(|N_T(y')| \geq |N_T(u)| + |Q| - 1 = |T| - 2 = n_2\).

Let \(q'\) be the vertex in \(Q\) which is not adjacent with \(y'\). If \(v_0q' \notin E(F_1)\), then \(v_0\) must be adjacent to \(q'\). (Otherwise \(T\) would contain \(C_4\) formed by \(\{v_0, y', q', u\}\)).

Now we have two new stars, namely \(S'_1 + n_1\) and \(S'_2 + n_2\), where \(V(S'_1 + n_1) = N_T[y']\) with \(y'\) as the center and \(V(S'_2 + n_2) = (S'_1 + n_1 \setminus \{y'\}) \cup \{y'\}\). If \(v_0u \in E(F_1)\), then we also have two new stars. The first one is \(S'_1 + n_1\) as in the previous case and the second one is \(S'_2 + n_2\) where \(V(S'_2 + n_2) = (S'_1 + n_1 \setminus \{y'\}) \cup \{u\}\) with \(v_0\) as the center. In case that \(y'\) is adjacent with all vertices in \(Q\) and \(v_0q \notin E(F_1)\), then the second star is \(S'_1 + n_2\) where \(V(S'_1 + n_2) = (S'_1 + n_1 \setminus \{y'\}) \cup \{q\}, q \in Q\) with \(v_0\) as the center. The fact that \(v_0q \in E(F_1)\) is guaranteed by Assumption 1.

Therefore, we have \(F_1 \supseteq S_1 + n_2 \cup S_1 + n_2\). Thus \(R(S_1 + n_2 \cup S_1 + n_2, C_4) \leq n_1 + n_2 + 3\). By Lemma 2.1 we have \(R(S_1 + n_2 \cup S_1 + n_2, C_4) \leq n_1 + n_2 + 3\).

We assume the theorem holds for every \(2 \leq r < k\). Let \(F_2\) be a graph of order \(\sum_{i=1}^k n_i + k + 1\). Suppose \(F_2\) contains no \(C_4\). We will show that \(F_2 \supseteq \bigcup_{i=1}^k S_1 + n_i\).

By induction hypothesis, \(F_2 \supseteq \bigcup_{i=1}^{k-1} S_1 + n_i\). Write \(B = F_2 \setminus \bigcup_{i=1}^{k-1} S_1 + n_i\) and \(T' = F_2[B]\). Thus \(|T'| = n_{k-1} + n_k + 3\). Since \(T'\) contains no \(C_4\) and its follows from the case \(k = 2\) that \(T'\) contains \(S_1 + n_{k-1} \cup S_1 + n_k\). Hence \(F_2\) contains \(\bigcup_{i=1}^k S_1 + n_i\). Thus we have \(R(\bigcup_{i=1}^k S_1 + n_i, C_4) \leq \bigcup_{i=1}^k n_i + k + 1\). On the other hand, we have \(R(\bigcup_{i=1}^k S_1 + n_i, C_4) \geq \sum_{i=1}^k n_i + k + 1\) (by Lemma 2.1). The assertion follows. \(\square\)

**Theorem 2.3.** Let \(n_i\) be natural number for \(i = 1, 2, \ldots, k\) and \(n_i \geq n_{i+1} \geq 3\) for every \(i\). If \(n_i\) is odd and \(2n_i + 1 \geq n_i\) for every \(i\), then \(R(\bigcup_{i=1}^k S_1 + n_i, W_4) = R(S_{n_k}, W_4) + \sum_{i=1}^k n_i \) for \(k \geq 1\).
Ramsey numbers for stars

PROOF. Let \( n_i \) be odd and \( 2n_{i+1} \geq n_i \) for every \( i \). Consider \( F \cong K_{-1 + \sum_{i=1}^{k} n_i} \cup K_{n_k - 1} \). Clearly, the graph \( F \) has order \(-2 + 2n_k + \sum_{i=1}^{k-1} n_i\), without containing \( \sum_{i=1}^{k} S_n \), and \( F \) contains no \( W_4 \). Hence,

\[
R(\bigcup_{i=1}^{k} S_n, W_4) \geq -1 + 2n_k + \sum_{i=1}^{k-1} n_i. \tag{1}
\]

To obtain the Ramsey number we use an induction on \( k \). For \( k = 1 \), we have

\[
R(S_n, W_4) = 2n - 1 \quad \text{(by Theorem 1)}.
\]

For \( k = 2 \), we show that

\[
R(S_n \cup S_n, W_4) = 2n - 1 + n_1 = R(S_n, W_4) + n_1.
\]

Let \( F_1 \) be a graph with \( |F_1| = 2n - 1 + n_1 = 2n_1 - 1 + 2n_2 - n_1 \). Assume that \( F_1 \) contains no \( W_4 \). We show that \( F_1 \) contains \( S_n \cup S_n \). Since \( 2n_2 \geq n_1 \), then \( |F_1| \geq 2n_1 - 1 \). By Theorem 1, \( F_1 \) contains \( S_n \). Write \( L = F_1 \setminus S_n \). Thus \(|L| = 2n_2 - 1\), such that \( L \) contains \( S_n \). Hence, \( F_1 \) contains \( S_n \cup S_n \). Therefore, \( R(S_n \cup S_n, W_4) \leq 2n_2 - 1 + n_1 \).

Suppose the theorem holds for every \( r < k \). Let \( F_2 \) be a graph of order \(-1 + 2n_k + \sum_{i=1}^{k-1} n_i\). Suppose \( F_2 \) contains no \( W_4 \). By the assumption, \( F_2 \) contains \( \bigcup_{i=1}^{k-1} S_n \). Let \( L' = F_2 \setminus \bigcup_{i=1}^{k-1} S_n \). Thus \(|L'| = 2n_k - 1\). Since \( F_2 \) contains no \( W_4 \), then by Theorem 1, \( L' \supset S_n \). Hence, \( F_2 \) contains \( \bigcup_{i=1}^{k} S_n \). Therefore, we have

\[
R(\bigcup_{i=1}^{k} S_n, W_4) = -1 + 2n_k + \sum_{i=1}^{k-1} n_i = R(S_n, W_4) + \sum_{i=1}^{k-1} n_i. \tag{2}
\]

\[ \square \]

References
