FIXED POINT THEOREMS FOR GENERALIZED
CONTRACTION MAPPINGS IN QUASI
\(a_b\)-METRIC SPACE

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Abstract

In this paper, we introduce quasi \(a_b\)-metric space as a generalization of a quasi \(b\)-metric space and prove the existence of some fixed point theorems, including its uniqueness; for several different contraction mappings on a quasi \(a_b\)-metric space. Particular examples are given to clarify the notion of a quasi \(a_b\)-metric space.

1. Introduction

Fixed point theory is one of the main topics in nonlinear analysis. The celebrated fixed point theorem was proved by Banach. The theorem is known as the Banach’s contraction fixed point theorem. Generalizations of this can be found in [1] and [2]. In this paper, we extend the concept of a quasi \(b\)-metric space by modifying the triangularity condition in the notion of a
quasi $b$-metric. The extension is called a quasi $αb$-metric. The concept of $b$-metric introduced by Bakhtin [3] and Czerwik [4] is applied to obtain some fixed point theorems for contraction mappings in a $b$-metric space. Furthermore, many authors proved fixed point theorems in a $b$-metric space ([5-7]), and the quasi $b$-metric space was usually used as a dislocated quasi $b$-metric space for fixed point theorems [8, 9]. Using this idea, we present generalization of some fixed point theorems of Banach type in some generalized $b$-metric spaces, especially in the quasi $αb$-metric space.

The purpose of this paper is to establish some fixed point theorems in a complete quasi $αb$-metric space using contraction mappings.

2. Preliminaries

**Definition 2.1** [3, 6]. Let $X$ be a nonempty set and let $b \geq 1$ be a given real number.

Let $d : X \times X \to [0, \infty)$ be a self-mapping on $X$ and for all $x, y, z \in X$, the following conditions be satisfied:

1. $d(x, y) = d(y, x) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq b(d(x, z) + d(z, y))$.

Then $d$ is called a $b$-metric on $X$ and $(X, d)$ is called a $b$-metric space. The pair $(X, d)$ is called a quasi $b$-metric space if (1) and (3) hold. Clearly, a $b$-metric is quasi $b$-metric. But the converse is not necessarily true.

**Example 2.2.** If $X = R$, then $d(x, y) = (x - y)^2$ defines a $b$-metric with $b = 2$.

If $X = R^+$, then

$$d(x, y) = \begin{cases} (2x + y)^2, & x \neq y, \\ 0, & x = y \end{cases}$$

defines quasi $b$-metric with $b = 2$. 
Definition 2.3. Let $X$ be a nonempty set. Let $0 \leq \alpha < 1$ and $b \geq 1$. Let a map $d : X \times X \to [0, \infty)$ satisfy the conditions:

1. $d(x, y) = d(y, x) = 0$ if and only if $x = y$;
2. $d(x, y) \leq \alpha d(y, x) + \frac{1}{2} b(d(x, z) + d(z, y))$ for all $x, y, z \in X$. (2.1)

Then $d$ is called a quasi $\alpha b$-metric on $X$ and $(X, d)$ is called a quasi $\alpha b$-metric space.

It is clear that a quasi $b$-metric is a quasi $\alpha b$-metric. The converse is not necessarily true as seen from the following examples.

Example 2.4. Let $X = \{0, 1, 2\}$. Define $d : X \times X \to R^+$ as follows:

$d(0, 0) = d(1, 1) = d(2, 2) = d(0, 2) = d(2, 1) = 0$, $d(1, 0) = 4$, $d(2, 0) = 1$, $d(0, 1) = 2$, and $d(1, 2) = 3$. It is clear that $d$ is quasi $\alpha b$-metric with $\alpha = \frac{1}{2}$ and $b = 4$, because $2 = d(0, 1) \leq \frac{1}{2} d(1, 0) + 2(d(0, 2) + d(2, 1))$, but $2 = d(0, 1) > c(d(0, 2) + d(2, 1))$ for every $c \geq 1$, so $d$ is not a quasi $b$-metric.

Example 2.5. Let $X = R$ and define $d : X \times X \to R^+$ by

$$d(x, y) = \begin{cases} 2x^2 + y^2, & x \neq y, \\ 0, & x = y. \end{cases}$$

For $x \neq y$, and every $z \in X$, we have

$$d(x, y) = 2x^2 + y^2 \leq \frac{5}{2} x^2 + 2y^2 + 3z^2$$

$$= \frac{1}{2} (2y^2 + x^2) + ((2x^2 + z^2) + (2z^2 + y^2))$$

$$= \frac{1}{2} d(y, x) + \frac{2}{2} (d(x, z) + d(z, y)).$$
Therefore,

\[ d(x, y) = \frac{1}{2} d(y, x) + \frac{2}{2} (d(x, z) + d(z, y)). \]

Hence, \( d \) is a quasi \( \alpha b \)-metric with \( \alpha = \frac{1}{2} \) and \( b = 2 \).

**Definition 2.6.** Let \( (X, d) \) be a quasi \( \alpha b \)-metric space. Then a sequence \( \{x_n\} \) in \( (X, d) \) is said to converge to \( x \in X \) if \( \lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0 \), we write \( \lim_{n \to \infty} x_n = x \).

**Definition 2.7.** Let \( \{x_n\} \) be a sequence in quasi \( \alpha b \)-metric space \( (X, d) \), \( \{x_n\} \) is called Cauchy sequence if

\[ \lim_{n,m \to \infty} d(x_n, x_m) = \lim_{m \to \infty} d(x_m, x_n) = 0. \]

**Definition 2.8.** A quasi \( \alpha b \)-metric space \( (X, d) \) is called complete if every Cauchy sequence in \( X \) converges in \( X \).

**Definition 2.9** [1, 2]. Let \( X \) be a nonempty set and let \( T \) be a self-mapping on \( X \). Then \( x \in X \) is called a fixed point of \( T \), if \( Tx = x \).

Let \( X \) be a nonempty set and \( T \) be a self-mapping on \( X \). Then for every \( x \in X \), we define \( TT^{n-1} x = T^n x \) with \( T^0 x = x \).

**Definition 2.10** [1]. Let \( (X, d) \) be a metric space, and let \( T \) be a self-mapping on \( X \). Then \( T \) is called a Banach contraction if there exists \( 0 < \lambda < 1 \) such that \( d(Tx, Ty) \leq \lambda d(x, y) \), for every \( x, y \in X \).

**Definition 2.11.** Let \( (X, d) \) and \( (Y, d) \) be quasi \( \alpha b \)-metric spaces. Then a map \( T : X \to Y \) is called continuous on \( X \) if every \( x, y \in X \) and \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( d(Tx, Ty) < \varepsilon \), whenever \( d(x, y) < \delta \).

### 3. Main Results

**Lemma 3.1.** Let \( (X, d) \) be a quasi \( \alpha b \)-metric space with \( 0 \leq \alpha < 1 \) and
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$b \geq 1$, and let \( \{x_n\} \) be a sequence in \( X \). If \( \{x_n\} \) converges to \( x \in X \), then \( x \) is unique.

**Proof.** It is straightforward.

**Theorem 3.2.** Let \((X, d)\) be a quasi \(\alpha\)-\(b\)-metric space with \(0 \leq \alpha < 1\) and \(b \geq 1\). Then every convergent sequence in \( X \) is a Cauchy sequence in \( X \).

**Proof.** Let \( \{x_n\} \) be a sequence in \( X \) and \( \{x_n\} \) converges to \( x \in X \). Then \( \lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0 \).

From (1), we obtain

\[
d(x_n, x_m) \leq \alpha d(x_m, x_n) + \frac{b}{2} (d(x_n, x) + d(x, x_m))
\]

\[
\leq \alpha (\alpha d(x_n, x_m)) + \frac{b}{2} (d(x_n, x) + d(x, x_n))
\]

\[
+ \frac{b}{2} (d(x_n, x) + d(x, x_m)).
\]

Thus

\[
(1 - \alpha^2) d(x_n, x_m) \leq \frac{ab}{2} (d(x_m, x) + d(x, x_n)) + \frac{b}{2} (d(x_n, x) + d(x, x_m)).
\]

Hence, we obtain

\[
d(x_n, x_m) \leq \frac{ab}{2} (d(x_m, x) + d(x, x_n)) + \frac{b}{2} (d(x_n, x) + d(x, x_m))
\]

\[
\frac{1 - \alpha^2}{1 - \alpha^2}.
\]

Similarly, we have

\[
d(x_m, x_n) \leq \frac{ab}{2} (d(x_n, x) + d(x, x_m)) + \frac{b}{2} (d(x_m, x) + d(x, x_n))
\]

\[
\frac{1 - \alpha^2}{1 - \alpha^2}.
\]

So for \( n, m \to \infty \), by using the convergence conditions, we have

\[
\lim_{n, m \to \infty} d(x_n, x_m) = \lim_{m, n \to \infty} d(x_m, x_n) = 0.
\]

Hence, \( \{x_n\} \) is a Cauchy sequence in \( X \).
Theorem 3.3. Let \((X, d)\) be a quasi \(\alpha b\)-metric space with \(0 \leq \alpha < 1\) and \(b \geq 1\), and let \(\{x_n\}\) be a sequence in \(X\) satisfying the conditions as follows:

(1) \(d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n)\), where \(0 < a < 1\);

(2) \(d(x_{n+1}, x_n) \leq cd(x_n, x_{n-1})\), where \(0 < c < 1\);

(3) \(ba + \alpha^2 < 1\) and \(bc + \alpha^2 < 1\).

Then \(\{x_n\}\) is a Cauchy sequence in \(X\).

Proof. From (2.1), we obtain

\[
d(x_n, x_{n+2}) \leq \alpha d(x_{n+2}, x_n) + \frac{b}{2} (d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}))
\]

\[
\leq \alpha \left[ ad(x_{n+2}, x_n) + \frac{b}{2} (d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)) \right]
\]

\[
+ \frac{b}{2} (d(x_{n+1}, x_n) + d(x_{n+1}, x_{n+2}))
\]

\[
\frac{1}{2} \alpha b (d(x_{n+2}, x_n) + d(x_{n+1}, x_n))
\]

\[
d(x_n, x_{n+2}) \leq \frac{\frac{1}{2} \alpha b (c^{n+1} + c^n) d(x_1, x_0) + \frac{b}{2} (a^n + a^{n+1}) d(x_0, x_1)}{1 - \alpha^2}.
\]

By (1) and (2), we get

\[
d(x_n, x_{n+2}) \leq \frac{\frac{1}{2} \alpha b (c^{n+1} + c^n) d(x_1, x_0) + \frac{b}{2} (a^n + a^{n+1}) d(x_0, x_1)}{1 - \alpha^2}.
\]

Since \(0 \leq \alpha < 1\), we have

\[
d(x_n, x_{n+2}) \leq \frac{b}{2(1 - \alpha^2)} [(c^{n+1} + c^n) d(x_1, x_0) + (a^n + a^{n+1}) d(x_0, x_1)].
\]

Further, since \(0 \leq \alpha < 1\), we have \(\frac{1}{1 - \alpha^2} > 1\) and by using \(b \geq 1\), we get

\[
\frac{b}{1 - \alpha^2} > 1.
\]
Thus

\[ d(x_n, x_{n+2}) \leq \frac{1}{2} \left[ \left( \frac{bc}{1 - \alpha^2} \right)^{n+1} + \left( \frac{bc}{1 - \alpha^2} \right)^n d(x_1, x_0) \right. \]

\[ + \left. \left( \frac{ba}{1 - \alpha^2} \right)^n + \left( \frac{ba}{1 - \alpha^2} \right)^{n+1} d(x_0, x_1) \right]. \]

Since \( \rho = \frac{bc}{1 - \alpha^2} \) and \( \gamma = \frac{ba}{1 - \alpha^2} \), we have

\[ d(x_n, x_{n+2}) \leq \frac{1}{2} \left[ (\rho^{n+1} + \rho^n) d(x_1, x_0) + (\gamma^n + \gamma^{n+1}) d(x_0, x_1) \right]. \]

By repeating this process for \( d(x_n, x_{n+3}) \), we get

\[ d(x_n, x_{n+3}) \leq \frac{\alpha b}{2} \left( \frac{e^{n+2} d(x_1, x_0) + b \alpha^{n+2} d(x_0, x_1)}{1 - \alpha^2} \right. \]

\[ + \left. \left( \frac{b}{2} \right)^2 (\alpha^n + a^n) d(x_0, x_1) \right) \]

\[ + \frac{2 \alpha \left( \frac{b}{2} \right)^2 (c^n + c^{n+1}) d(x_1, x_0) \}}{(1 - \alpha^2)^2}. \]

Since \( 0 \leq \alpha < 1 \), we have

\[ d(x_n, x_{n+3}) \leq \frac{\alpha b}{2} \left( \frac{e^{n+2} d(x_1, x_0) + b \alpha^{n+2} d(x_0, x_1)}{1 - \alpha^2} \right. \]

\[ + \left. \left( \frac{b}{2} \right)^2 (\alpha^n + a^n) d(x_0, x_1) \right) \]

\[ + \frac{2 \alpha \left( \frac{b}{2} \right)^2 (c^n + c^{n+1}) d(x_1, x_0) \}}{(1 - \alpha^2)^2}. \]
Further, since $0 \leq \alpha < 1$, we have $\frac{1}{1-\alpha^2} > 1$ and by using $b \geq 1$, we get $\frac{b}{1-\alpha^2} > 1$ and $\left(\frac{b}{1-\alpha^2}\right)^2 > 1$. Therefore,

$$d(x_n, x_{n+3}) \leq \frac{1}{2} \left[ \left(\frac{bc}{1-\alpha^2}\right)^n d(x_1, x_0) + \left(\frac{ba}{1-\alpha^2}\right)^n d(x_0, x_1) \right]$$

$$+ \frac{1}{2} \left[ \left(\frac{ba}{1-\alpha^2}\right)^{n+1} d(x_0, x_1) + \left(\frac{bc}{1-\alpha^2}\right)^{n+1} d(x_1, x_0) \right]$$

$$+ \frac{1}{2} \left[ \left(\frac{ba}{1-\alpha^2}\right)^n d(x_0, x_1) + \left(\frac{bc}{1-\alpha^2}\right)^n d(x_1, x_0) \right].$$

Since $\rho = \frac{bc}{1-\alpha^2}$ and $\gamma = \frac{ba}{1-\alpha^2}$, we have

$$d(x_n, x_{n+3}) \leq \frac{1}{2} \left[ (\rho^n + \rho^{n+1} + \rho^{n+2}) d(x_1, x_0) \right]$$

$$+ [(\gamma^n + \gamma^{n+1} + \gamma^{n+2}) d(x_0, x_1)].$$

In general, for $d(x_n, x_{n+k})$, we obtain

$$d(x_n, x_{n+k}) \leq \frac{1}{2} \left[ (\rho^n + \rho^{n+1} + \rho^{n+2} + \cdots + \rho^{n+k-1}) d(x_1, x_0) \right]$$

$$+ (\gamma^n + \gamma^{n+1} + \gamma^{n+2} + \cdots + \gamma^{n+k-1}) d(x_0, x_1)].$$

So, for $m > n$, we have

$$d(x_n, x_m) \leq \frac{1}{2} \left[ (\rho^n + \rho^{n+1} + \rho^{n+2} + \cdots + \rho^{m-1}) d(x_1, x_0) \right]$$

$$+ (\gamma^n + \gamma^{n+1} + \gamma^{n+2} + \cdots + \gamma^{m-1}) d(x_0, x_1)]$$

$$= \frac{1}{2} \rho^n \left[ 1 + \rho + \rho^2 + \cdots + \rho^{m-n-1} \right] d(x_1, x_0)$$

$$+ \gamma^n \left[ 1 + \gamma + \gamma^2 + \cdots + \gamma^{m-n-1} \right] d(x_0, x_1)].$$
Since $ba + \alpha^2 < 1$, $bc + \alpha^2 < 1$, $\gamma = \frac{ba}{1-\alpha^2} < 1$ and $\rho = \frac{bc}{1-\alpha^2} < 1$.

Thus for $m \to \infty$, we get $d(x_n, x_m) \leq \frac{\rho^n}{2(1-\rho)} + \frac{\gamma^n}{2(1-\gamma)}$ and for $n \to \infty$, we get $d(x_n, x_m) \to 0$.

Similarly, we get

$$d(x_{n+2}, x_n) \leq \frac{1}{2} \alpha b(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})) + \frac{b}{2}(d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n))$$

$$d(x_{n+2}, x_n) \leq \frac{b}{2}(c^{n+1} + c^n)d(x_1, x_0) + \frac{1}{2} \alpha b(a^n + a^{n+1})d(x_0, x_1)$$

$$d(x_{n+2}, x_n) \leq \frac{b}{2}(c^{n+1} + c^n)d(x_0, x_0) + \frac{1}{2} b(a^n + a^{n+1})d(x_0, x_1)$$

Since $0 \leq \alpha < 1$, we have $\frac{1}{1-\alpha^2} > 1$, and by using $b \geq 1$, we get $\frac{b}{1-\alpha^2} > 1$.

Thus, we get

$$d(x_{n+2}, x_n) \leq \frac{1}{2} \left[ \left( \frac{bc}{1-\alpha^2} \right)^{n+1} + \left( \frac{bc}{1-\alpha^2} \right)^n \right]d(x_1, x_0)$$

$$+ \left[ \left( \frac{ba}{1-\alpha^2} \right)^n + \left( \frac{ba}{1-\alpha^2} \right)^{n+1} \right]d(x_0, x_1)$$

$$d(x_{n+2}, x_n) \leq \frac{1}{2} \left[ (\rho^{n+1} + \rho^n)d(x_1, x_0) + (\gamma^n + \gamma^{n+1})d(x_0, x_1) \right].$$
Similarly, we get

\[
d(x_{n+3}, x_n) \leq \frac{\alpha b}{2} d(x_0, x_1) + \frac{b}{2} e^{n+2} d(x_1, x_0) - \frac{1}{1 - \alpha^2} \left( \left( \frac{\alpha b}{2} \right)^2 + \left( \frac{b}{2} \right)^2 \right) (e^{n+1} + e^n) d(x_1, x_0) + 2\alpha \left( \frac{b}{2} \right)^2 (a^n + a^{n+1}) d(x_0, x_1)
\]

Since \( 0 \leq \alpha < 1 \), we have

\[
d(x_{n+3}, x_n) \leq \frac{b}{2} d(x_0, x_1) a^{n+2} + \frac{b}{2} e^{n+2} d(x_1, x_0) - \frac{1}{1 - \alpha^2} \left( \left( \frac{b}{2} \right)^2 + \left( \frac{b}{2} \right)^2 \right) (e^{n+1} + e^n) d(x_1, x_0) + 2\left( \frac{b}{2} \right)^2 (a^n + a^{n+1}) d(x_0, x_1)
\]

Since \( 0 \leq \alpha < 1 \), we have \( \frac{1}{1 - \alpha^2} > 1 \), and by using \( b \geq 1 \), we get \( \frac{b}{1 - \alpha^2} > 1 \)

and \( \left( \frac{b}{1 - \alpha^2} \right)^2 > 1 \). Thus

\[
d(x_{n+3}, x_n) \leq \frac{1}{2} \left[ \left( \frac{ba}{1 - \alpha^2} \right)^{n+2} d(x_0, x_1) + \left( \frac{bc}{1 - \alpha^2} \right)^{n+2} d(x_1, x_0) \right] + \frac{1}{2} \left[ \left( \frac{bc}{1 - \alpha^2} \right)^{n+1} + \left( \frac{bc}{1 - \alpha^2} \right)^n d(x_1, x_0) \right] + \left( \frac{ba}{1 - \alpha^2} \right)^n + \left( \frac{ba}{1 - \alpha^2} \right)^{n+1} d(x_0, x_1)
\]
Since $\rho = \frac{bc}{1 - \alpha^2}$ and $\gamma = \frac{ba}{1 - \alpha^2}$, we have

$$(x_{n+3}, x_n) \leq \frac{1}{2} [\gamma^{n+2}d(x_0, x_1) + \rho^{n+2}d(x_1, x_0)]$$

$$+ \frac{1}{2} [(\rho^{n+1} + \rho^n)d(x_1, x_0) + (\gamma^n + \gamma^{n+1})d(x_0, x_1)].$$

Therefore,

$$d(x_{n+3}, x_n) \leq \frac{1}{2} [(\gamma^n + \gamma^{n+1} + \gamma^{n+2})d(x_0, x_1)$$

$$+ (\rho^n + \rho^{n+1} + \rho^{n+2})d(x_1, x_0)].$$

In general, for $d(x_{n+k}, x_n)$, we obtain

$$d(x_{n+k}, x_n) \leq \frac{1}{2} [(\gamma^n + \gamma^{n+1} + \gamma^{n+2} + \cdots + \gamma^{n+k-1})d(x_0, x_1)$$

$$+ (\rho^n + \rho^{n+1} + \rho^{n+2} + \cdots + \rho^{n+k-1})d(x_1, x_0)].$$

So, for $n > m$, we have

$$d(x_m, x_n) \leq \frac{1}{2} [(\gamma^n + \gamma^{n+1} + \gamma^{n+2} + \cdots + \gamma^{m-1})d(x_0, x_1)$$

$$+ (\rho^n + \rho^{n+1} + \rho^{n+2} + \cdots + \rho^{m-1})d(x_1, x_0)]$$

$$= \frac{1}{2} \rho^n[(1 + \rho + \rho^2 + \cdots + \rho^{m-n-1})d(x_1, x_0)$$

$$+ \gamma^n(1 + \gamma + \gamma^2 + \cdots + \gamma^{m-n-1})d(x_0, x_1)].$$

Since $ba + \alpha^2 < 1$ and $bc + \alpha^2 < 1$, $\gamma = \frac{ba}{1 - \alpha^2} < 1$ and $\rho = \frac{bc}{1 - \alpha^2} < 1$.

Thus

$$\lim_{n,m \to \infty} d(x_n, x_m) = \lim_{m,n \to \infty} d(x_m, x_n) = 0.$$

Hence, $\{x_n\}$ is a Cauchy sequence in $X$. 
Lemma 3.4. Let \((X, d)\) be a quasi \(ab\)-metric space with \(0 \leq \alpha < 1\) and \(b \geq 1\), and let \(T\) be a self-mapping on \(X\) that satisfies the condition of Banach contraction \(d(Tx, Ty) \leq \lambda d(x, y)\) for \(x, y \in X\), \(0 < \lambda < 1\). Then \(T\) is continuous on \(X\).

Proof. It is easy.

Theorem 3.5. Let \((X, d)\) be a complete quasi \(ab\)-metric space with \(0 \leq \alpha < 1\) and \(b \geq 1\), and let \(T\) be a self-mapping on \(X\) that satisfies the condition of Banach contraction as follows \(d(Tx, Ty) \leq \lambda d(x, y)\) for \(x, y \in X\), \(0 < \lambda < 1\) and \(b\lambda + \alpha^2 < 1\). Then \(T\) has a unique fixed point in \(X\).

Proof. Let \(\{x_n\}\) be a sequence in \(X\) such that \(x_{n+1} = Tx_n\) for \(n = 0, 1, 2, 3, \ldots\). From the condition of Banach contraction, we get

\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \lambda(d(x_{n-1}, x_n)) \quad \text{and} \quad d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \lambda(d(x_n, x_{n-1})).
\]

Since \(b\lambda + \alpha^2 < 1\), using Theorem 3.3, we have \(\{x_n\}\) as a Cauchy sequence in the complete quasi \(ab\)-metric space \((X, d)\). So there exists \(x^* \in X\) such that \(\lim_{n \to \infty} x_n = x^*\). Now we show that \(x^*\) is a fixed point of \(T\).

By Lemma 3.4, \(T\) is continuous on \(X\). Therefore,

\[
Tx^* = T \lim_{n \to \infty} x_n = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = x^*.
\]

Thus \(Tx^* = x^*\). Hence, \(x^*\) is a fixed point of \(T\).

That the fixed point is unique is easy to establish.

Theorem 3.6. Let \((X, d)\) be a complete quasi \(ab\)-metric space with \(0 \leq \alpha < 1\) and \(b \geq 1\). Let \(T\) be a continuous self-mapping on \(X\) that satisfies the following condition:
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\[ d(Tx, Ty) \leq \frac{d(y, Ty)d(Ty, y)}{1 + p(d(y, Ty) + d(y, Ty))} + \frac{qd(x, y)d(y, x)}{1 + d(x, y) + d(y, x)}. \quad (3.1) \]

For \( x, y \in X \), and \( p > 1, \ 0 < q < 1 \) with \( (1 - q)p > 1 \) and \( bpq < (1 - \alpha^2)(p - 1) \). Then \( T \) has a unique fixed point in \( X \).

**Proof.** Let \( x_0 \in X \). Define a sequence \( \{x_n\} \) by \( x_{n+1} = Tx_n \) and \( x_n \neq x_{n+1} \) for every \( n = 0, 1, 2, 3, \ldots \).

Using (3.1), we have

\[ d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \]

\[ \leq \frac{d(x_n, x_{n+1})d(x_{n+1}, x_n)}{1 + p(d(x_n, x_{n+1}) + d(x_{n+1}, x_n))} \]

\[ + \frac{qd(x_{n-1}, x_n)d(x_n, x_{n-1})}{1 + d(x_{n-1}, x_n) + d(x_n, x_{n-1})} \]

\[ \leq \frac{d(x_n, x_{n+1})d(x_{n+1}, x_n)}{pd(x_{n+1}, x_n)} + \frac{qd(x_{n-1}, x_n)d(x_n, x_{n-1})}{d(x_n, x_{n-1})} \]

\[ = \frac{d(x_n, x_{n+1})}{p} + qd(x_{n-1}, x_n). \]

Thus

\[ d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n). \]

Similarly,

\[ d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}), \text{ where } \lambda = \frac{pq}{p - 1}. \]

Since \( (1 - q)p > 1 \), \( \lambda = \frac{pq}{p - 1} < 1 \). Further, since \( bpq < (1 - \alpha^2)(p - 1) \),

we have \( b\lambda + \alpha^2 < 1 \).
Now from Theorem 3.3, it follows that \( \{x_n\} \) is a Cauchy sequence in a complete quasi \( \alpha \beta \)-metric space \( X \). So there exists \( x^* \in X \) such that \( \lim_{n\to\infty} x_n = x^* \). Now, we show that \( x^* \) is a fixed point of \( T \). By the continuity of \( T \), we have

\[
T x^* = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} T x_n = \lim_{n\to\infty} x_{n+1} = x^*.
\]

Thus \( T x^* = x^* \). Hence, \( x^* \) is a fixed point of \( T \).

Next, we show that \( x^* \) is unique.

Suppose there exists \( y^* \in X \), such that \( Ty^* = y^* \). Then from (3.1), we get

\[
d(x^*, y^*) = d(T x^*, T y^*) \leq \frac{d(y^*, Ty^*)d(T y^*, y^*)}{1 + p(d(y^*, Ty^*)d(T y^*, y^*))} + \frac{qd(x^*, y^*)d(y^*, x^*)}{1 + d(x^*, y^*) + d(y^*, x^*)}.
\]

Since \( d(y^*, y^*) = 0 \) and because \( x^* \neq y^* \), we have \( d(y^*, x^*) \neq 0 \), and obtain

\[
d(x^*, y^*) \leq \frac{d(y^*, y^*)d(y^*, y^*)}{1 + p(d(y^*, y^*) + d(y^*, y^*)} + \frac{qd(x^*, y^*)d(y^*, x^*)}{d(y^*, x^*)}.
\]

Since \( 0 < q < 1 \), \( d(x^*, y^*) = 0 \). Similarly, we have \( d(y^*, x^*) = 0 \). Thus \( x^* = y^* \).

**Theorem 3.7.** Let \((X, d)\) be a complete quasi \( \alpha \beta \)-metric space with \( 0 \leq \alpha < 1 \) and \( b \geq 1 \). Let \( T \) be a continuous self-mapping on \( X \) satisfying the following condition:
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\[ d(Tx, Ty) \leq \frac{k[d(Ty, y)d(y, Tx) + d(x, Ty)d(x, Tx)]}{1 + d(y, Tx) + d(x, Ty)}, \]

\( \forall x, y \in X, 0 < k < 1 \) and \( kb + \alpha^2 < 1 \). Then \( T \) has a unique fixed point in \( X \).

**Proof.** Let \( \{x_n\} \) be a sequence in \( X \) such that \( x_{n+1} = Tx_n \) and \( x_n \neq x_{n+1} \) for \( n = 0, 1, 2, 3, \ldots \),

\[ d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \]

\[ \leq \frac{k[d(x_{n+1}, x_n)d(x_n, x_n) + d(x_{n-1}, x_{n+1})d(x_{n-1}, x_n)]}{1 + d(x_n, x_n) + d(x_{n-1}, x_{n+1})} \]

\[ = \frac{kd(x_{n-1}, x_{n+1})d(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_{n+1})} \]

\[ \leq \frac{kd(x_{n-1}, x_{n+1})d(x_{n-1}, x_n)}{d(x_{n-1}, x_{n+1})} \]

\[ = kd(x_{n-1}, x_n). \]

Therefore, we get \( d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n). \) Next

\[ d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \]

\[ \leq \frac{k[d(x_n, x_{n-1})d(x_{n-1}, x_{n+1}) + d(x_n, x_n)d(x_n, x_{n+1})]}{1 + d(x_{n-1}, x_{n+1}) + d(x_n, x_n)} \]

\[ = \frac{k[d(x_n, x_{n-1})d(x_{n-1}, x_{n+1})]}{1 + d(x_{n-1}, x_{n+1})} \]

\[ \leq \frac{k[d(x_n, x_{n-1})d(x_{n-1}, x_{n+1})]}{d(x_{n-1}, x_{n+1})} = kd(x_n, x_{n-1}). \]

So, we get

\[ d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1}). \]
Since \( kb + \alpha^2 < 1 \), using Theorem 3.3, \( \{x_n\} \) is a Cauchy sequence in complete quasi \( \alpha \beta \)-metric space \( (X, d) \). Therefore, there exists an \( x^* \in X \) such that \( \lim_{n \to \infty} x_n = x^* \). Now we show that \( x^* \) is a fixed point of \( T \).

Since \( T \) is continuous on \( X \), we obtain

\[
T x^* = T \lim_{n \to \infty} x_n = \lim_{n \to \infty} T x_n = \lim_{n \to \infty} x_{n+1} = x^*.
\]

Thus \( T x^* = x^* \). Hence, \( x^* \) is a fixed point of \( T \).

**Uniqueness.** Suppose there exists \( y^* \in X \) such that \( T y^* = y^* \). Then

\[
d(x^*, y^*) = d(Tx^*, Ty^*)
\]

\[
\leq \frac{k[d(Ty^*, y^*)d(y^*, Tx^*) + d(x^*, Ty^*)d(x^*, Tx^*)]}{1 + d(y^*, Tx^*) + d(x^*, Ty^*)}
\]

\[
= \frac{k[d(y^*, y^*)d(y^*, Tx^*) + d(x^*, Ty^*)d(x^*, x^*)]}{1 + d(y^*, Tx^*) + d(x^*, Ty^*)} = 0.
\]

Thus, we get \( d(x^*, y^*) = 0 \). Now

\[
d(y^*, x^*) = d(Ty^*, Tx^*)
\]

\[
\leq \frac{k[d(Tx^*, x^*)d(x^*, Ty^*) + d(y^*, Tx^*)d(y^*, Ty^*)]}{1 + d(x^*, Ty^*) + d(y^*, Tx^*)}
\]

\[
= \frac{k[d(x^*, x^*)d(x^*, Ty^*) + d(y^*, Tx^*)d(y^*, y^*)]}{1 + d(x^*, Ty^*) + d(y^*, Tx^*)} = 0.
\]

Thus, we get \( d(y^*, x^*) = 0 \). From \( d(x^*, y^*) = 0 \) and \( d(y^*, x^*) = 0 \), we obtain \( x^* = y^* \). Hence, fixed point of \( T \) is unique.
Theorem 3.8. Let \((X, d)\) be a complete quasi \(\alpha b\)-metric space with \(0 \leq \alpha < 1\) and \(b \geq 1\). Let \(T\) be a continuous self-mapping on \(X\) that satisfies the following conditions:

\[
d(Tx, Ty) \leq \lambda_1 d(x, y) + \frac{\lambda_2 [d(Tx, x)d(y, Tx) + d(y, Ty)d(x, Ty)]}{1 + d(x, Ty) + d(y, Tx)},
\]

for every \(x, y \in X\), \(\lambda_1, \lambda_2 > 0\), \(0 < \lambda_1 + \lambda_2 < 1\) and \(b \frac{\lambda_1}{1 - \lambda_2} + \alpha^2 < 1\). (3.2)

Then \(T\) has a unique fixed point in \(X\).

**Proof.** Let \(x_0 \in X\). Define a sequence \(\{x_n\}\) as \(x_{n+1} = Tx_n = T^n x_0\) and \(x_n \neq x_{n+1}\) for \(n = 0, 1, 2, 3, \ldots\). Thus, from (3.2), we get

\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \lambda_1 d(x_{n-1}, x_n) + \frac{\lambda_2 [d(x_n, x_{n-1})d(x_{n-1}, x_n) + d(x_n, x_{n+1})d(x_{n-1}, x_{n+1})]}{1 + d(x_{n-1}, x_{n+1}) + d(x_n, x_{n-1})}
= \lambda_1 d(x_{n-1}, x_n) + \frac{\lambda_2 d(x_n, x_{n+1})d(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_{n+1})}
\leq \lambda_1 d(x_{n-1}, x_n) + \frac{\lambda_2 d(x_n, x_{n+1})d(x_n, x_{n-1})}{d(x_n, x_{n-1})}
= \lambda_1 d(x_{n-1}, x_n) + \lambda_2 d(x_n, x_{n+1}).
\]

Hence, \(d(x_n, x_{n+1}) \leq \frac{\lambda_1}{1 - \lambda_2} d(x_{n-1}, x_n)\). Next

\[
d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \lambda_1 d(x_n, x_{n-1}) + \frac{\lambda_2 [d(x_{n+1}, x_n)d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)d(x_n, x_{n+1})]}{1 + d(x_n, x_{n+1}) + d(x_{n-1}, x_{n+1})}
= \lambda_1 d(x_n, x_{n-1}) + \frac{\lambda_2 d(x_{n+1}, x_n)d(x_{n-1}, x_{n+1})}{1 + d(x_{n-1}, x_{n+1})}.
\]
\[
\leq \lambda_1 d(x_n, x_{n-1}) + \lambda_2 d(x_{n+1}, x_n) d(x_{n-1}, x_{n+1})
\]

\[
= \lambda_1 d(x_n, x_{n-1}) + \lambda_2 (x_{n+1}, x_n).
\]

So we get \( d(x_{n+1}, x_n) \leq \frac{\lambda_1}{1 - \lambda_2} d(x_n, x_{n-1}) \). Since \( \lambda_1, \lambda_2 > 0 \) and \( 0 < \lambda_1 + \lambda_2 < 1 \), we get \( d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \) and \( d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}) \), where \( \lambda = \frac{\lambda_1}{1 - \lambda_2} < 1 \). Further, since \( b = \frac{\lambda_1}{1 - \lambda_2} + \alpha^2 < 1 \), by using Theorem 3.3, we get that \( \{x_n\} \) is a Cauchy sequence in a complete quasi \( \alpha b \)-metric space \((X, d)\). So there exists \( x^* \in X \) such that \( \lim_{n \to \infty} d(x_n, x^*) = \lim_{n \to \infty} d(x^*, x_n) = 0 \).

Now, we show that \( x^* \) is a fixed point of \( T \).

Since \( T \) is continuous on \( X \), we obtain

\[
Tx^* = T \lim_{n \to \infty} x_n = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = x^*.
\]

Thus \( Tx^* = x^* \). Hence, \( x^* \) is a fixed point of \( T \).

**Uniqueness.** Suppose there exists \( y^* \in X \) such that \( Ty^* = y^* \). Then

\[
d(x^*, y^*) = d(Tx^*, Ty^*) \leq \lambda_1 d(x^*, y^*)
\]

\[
+ \lambda_2 [d(Tx^*, x^*) d(y^*, Tx^*) + d(y^*, Ty^*) d(x^*, Ty^*)] 
\]

\[
= \lambda_1 d(x^*, y^*) + \lambda_2 [d(x^*, x^*) d(y^*, x^*) + d(y^*, y^*) d(x^*, y^*)] 
\]

\[
= \lambda_1 d(x^*, y^*).
\]
Therefore, $d(x^*, y^*) \leq \lambda_1 d(x^*, y^*)$, is possible if $d(x^*, y^*) = 0$, because $\lambda_1 > 0$. Now

$$d(y^*, x^*) = d(Ty^*, Tx^*) \leq \lambda_1 d(y^*, x^*)$$

$$+ \frac{\lambda_2 [d(Ty^*, y^*)d(x^*, Ty^*) + d(x^*, Tx^*)d(y^*, Tx^*)]}{1 + d(y^*, Tx^*) + d(x^*, Ty^*)}$$

$$= \lambda_1 d(y^*, x^*) + \frac{\lambda_2 [d(y^*, y^*)d(x^*, y^*) + d(x^*, x^*)d(y^*, x^*)]}{1 + d(y^*, x^*) + d(x^*, y^*)}$$

$$= \lambda_1 d(y^*, x^*).$$

Thus, $d(y^*, x^*) \leq \lambda_1 d(y^*, x^*)$ is possible if $d(y^*, x^*) = 0$, because $\lambda_1 > 0$.

From $d(x^*, y^*) = 0$ and $d(y^*, x^*) = 0$, we have $x^* = y^*$.

Hence, we conclude that the fixed point of $T$ is unique.

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References


