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Convolution and Correlation Theorems for Wigner-Ville Distribution Associated with Linear Canonical Transform

Mawardi Bahri
Department of Mathematics
Hasanuddin University
Makassar 90245, Indonesia
Email: mawardibahri@gmail.com

Ryuichi Ashino
Division of Mathematical Sciences
Osaka Kyoiku University
Osaka 582-8582, Japan
Email: ashino@cc.osaka-kyoiku.ac.jp

***Abstract*—Generalized convolution and correlation theorems for the Wigner-Ville distribution (transform) associated with linear canonical transform (WVD-LCT) are established. The proposed theorems are modified forms of the convolution and correlation theorems of the linear canonical transform and classical Wigner-Ville distribution.**

Keywords: Wigner-Ville distribution, linear canonical transform, convolution, correlation, modulation

AMS Subject Classification: 11R52, 42C40

I. Introduction

In recent years, generalized convolution and correlation theorems for the linear canonical transform and their applications has attracted the serious attention of researchers. For some recent works and surveys on the subject we refer the reader to [3], [10], [13], [15], [16], [19], [20] and the references therein.

In [17], the generalization of the Wigner-Ville transform to the LCT domains so-called the Wigner-Ville distribution associated with linear canonical transform (WVD-LCT) was presented. The application of the WVD-LCT in the estimating of parameters for QFM signals was proposed in [18] and its uncertainty principle was recently studied in [9]. Some fundamental properties of generalized transform are already investigated such as the reversibility property, linear property, shift property and other properties. However, as far as we know, the general modulation property, convolution and correlation theorems, similar to the version of the LCT and Wigner-Ville distribution, are still not established or defined in the WVD-LCT domains.

Therefore, in the present work, we begin with introducing the WVD-LCT. The generalized transform is obtained by substituting the Fourier kernel with the LCT kernel in the WVT definition. In this paper, we first have established the various properties of the WVD-LCT such as translation, modulation, multiplication, and Moyal's formula. We then derive convolution and correlation theorems for the WVD-LCT, which are modified forms of the convolution and correlation theorems of the linear canonical transform and classical Wigner-Ville distribution. The results are very important for their applications in digital signal and image processing.

The paper is organized as follows: Section II briefly reviews the preliminaries about the LCT and its properties which will be needed in the sequel. The basic properties of the classical Wigner-Ville transform (WVT) are presented in this section. The construction of the WVD-LCT is studied in Section III; its important properties are also discussed in this section. Section IV provides our main result, that is, we establish new construction of convolution and correlation in the WVD-LCT domain. Some conclusions are drawn in Section V.

II. Preliminaries

To simplify our discussion, we start with some notations and definitions, which are used in the next section. In this paper, we deal with complex valued functions, which are usually called signals in the engineering community. We use the following convention to represent matrices:

$$A = (a, b; c, d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

For $1 \leq p < +\infty$, the space $L^p(\mathbb{R})$ consists of all complex-valued functions on $\mathbb{R}$ that satisfy

$$\int_{\mathbb{R}} |f(x)|^p \, dx < +\infty.$$

We define the L^p norm of f by

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p}.$$

Then $L^p(\mathbb{R})$ is a Banach space with the L^p norm. In the case of $p = 2$, we define the L^2 inner product of f and g by

$$(f,\, g) = \int_{\mathbb{R}} f(x)\overline{g(x)} \, dx,$$

and the L^2 norm by $\|f\| = \sqrt{(f,\, f)}$. Here $\bar{g}$ is the complex conjugate of the function g. Then $L^2(\mathbb{R})$ is a Hilbert space with the L^2 inner product.

A. Linear Canonical Transform

The linear canonical transform (LCT) is firstly proposed by Moshinsky and Collins [4], [8]. Here we briefly introduce

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the LCT definition. Hereafter we choose the branch of $\sqrt{z} = \sqrt{|z|}\, e^{i\theta/2}$, where $\theta = \operatorname{Arg} z$.

Definition 1 (LCT)**.** *Let $A = (a, b; c, d) \in \mathbb{R}^{2\times 2}$ be a matrix parameter satisfying* $\det A = ad - bc = 1$. *The LCT of a signal $f \in L^1(\mathbb{R})$ is defined by*

$$L_A\{f\}(\omega) = \begin{cases} \displaystyle\int_{\mathbb{R}} f(x) K_A(x,\omega)\, dx, b \neq 0 \\ \sqrt{d} e^{i\frac{cd}{2}\omega^2} f(d\omega),\ b = 0, \end{cases} \tag{1}$$

where $K_A(x,\omega)$ is so-called kernel of the LCT given by

$$K_A(x,\omega) = \frac{1}{\sqrt{2\pi b i}} e^{i\frac{1}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2\right)}.$$

It is not difficult to check that the LCT kernel mentioned above has the following important property

$$K_{A^{-1}}(x,\omega) = \frac{1}{\sqrt{-2\pi b i}} e^{-i\frac{1}{2}\left(\frac{a}{b}x^2 - \frac{2}{b}x\omega + \frac{d}{b}\omega^2\right)}.$$

From the definition of the LCT, when $b = 0$, we can see easily that the LCT of a signal is essentially a chirp multiplication. Therefore, in this work we always assume $b \neq 0$.

As a special case, when $A = (0, 1; -1, 0)$, the LCT (1) is nothing but the Fourier transform (FT).

The inverse LCT of a signal $g \in L^1(\mathbb{R})$ is given by

$$L_A^{-1}\{g\}(x) = \int_{\mathbb{R}} g(\omega)\, K_{A^{-1}}(\omega, x)\, d\omega, \tag{2}$$

where $A^{-1} = (d, -b; -c, a)$. Or, equivalently,

$$f(x) = \frac{1}{\sqrt{-2\pi b i}} \int_{\mathbb{R}} L_A\{f\}(\omega)\, e^{i\frac{1}{2}\left(\frac{-d}{b}\omega^2 + \frac{2}{b}x\omega - \frac{a}{b}x^2\right)}\, d\omega. \tag{3}$$

It is not difficult to see that the relationship between LCT and the FT is given by

$$L_A\{f\}(\omega) = \frac{1}{\sqrt{ib}} e^{\frac{id}{2b}\omega^2} \mathcal{F}\{e^{\frac{ia}{2b}x^2} f(x)\} \left(\frac{\omega}{b}\right), \tag{4}$$

where $\mathcal{F}\{f\}(\omega) = \hat{f}(\omega)$ is the FT of the function $f \in L^2(\mathbb{R})$ defined by (see Bracewell [7])

$$\mathcal{F}\{f\}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x}\, dx. \tag{5}$$

An important property of the LCT is Parseval's formula [3], which will be used to establish Moyal's formula for the WVD-LCT. That is,

$$(f, g) = (L_A\{f\}, L_A\{g\}). \tag{6}$$

In particular, for $f = g$, we obtain Plancherel's formula for the LCT as

$$\|f\|^2 = \|L_A\{f\}\|^2. \tag{7}$$

B. *Wigner-Ville Transform (Distribution)*

In this subsection we shortly recall the definition of the classical Wigner-Ville transform (WVT) or the Wigner-Ville distribution (WVD) and some of its properties. The detailed and comprehensive view can be found in [1], [2], [6], [12] and the references therein.

Definition 2 (Wigner-Ville transform)**.** *For $f,\ g \in L^2(\mathbb{R})$, the cross Wigner-Ville transform (distribution) of f and g is defined by*

$$\mathcal{W}_{f,g}(t,\omega) = \int_{\mathbb{R}} f\Big(t + \frac{x}{2}\Big) \overline{g\Big(t - \frac{x}{2}\Big)}\, e^{-i\omega x}\, dx. \tag{8}$$

Define the translation operator τ_a by $\tau_k f(x) = f(x-k)$ and the modulation operator M_{ω_0} by $M_{\omega_0} f(x) = e^{i\omega_0 x} f(x)$. Let $f,\, g \in L^2(\mathbb{R})$ be two signals. Some fundamental properties of the classical Wigner-Ville transform (WVT) are summarized as follows:

a. Complex conjugation
$$\overline{\mathcal{W}_{f,g}(t,\omega)} = \mathcal{W}_{g,f}(t,\omega).$$

b. Time marginal
$$\frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{W}_{f,g}(t,\omega)\, d\omega = |f(t)|^2.$$

c. Frequency marginal
$$\frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{W}_{f,g}(t,\omega)\, d\omega = |\hat{f}(\omega)|^2.$$

d. Energy distribution
$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{W}_{f,g}(t,\omega)\, d\omega\, dt = \|f\|^2 = \|\widehat{f}\,\|^2.$$

e. Translation/Shift
$$\mathcal{W}_{\tau_k f, \tau_k g}(t,\omega) = \mathcal{W}_{f,g}(t-k,\omega).$$

f. Modulation
$$\mathcal{W}_{M_{\omega_0} f, M_{\omega_0} g}(t,\omega) = \mathcal{W}_{f,g}(t,\omega-\omega_0).$$

g. Moyal's formula
$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{W}_f(t,\omega)\overline{\mathcal{W}_g(t,\omega)}\, d\omega\, dt = |(f,g)|^2.$$

h. Reconstruction formula
$$f(t) = \frac{1}{2\pi\overline{g(0)}}\int_{\mathbb{R}} \mathcal{W}_{f,g}\Big(\frac{t}{2},\omega\Big) e^{i\omega t}\, d\omega,$$
provided $\overline{g(0)} \neq 0$.

III. Wigner-Ville Distribution Associated with Linear Canonical Transform (WVD-LCT)

In [5], we studied the two-dimensional quaternion Wigner-Ville distribution (QWVD). It can be regarded an extension of the WVD using the quaternion algebra. On the other hand, the Wigner-Ville distribution associated with linear canonical transform (WVD-LCT) is a generalization of the WVD to the LCT domain. It was recently introduced by Bai et al. [17]. In this section, we will investigate more properties of the WVD-LCT which have not been explicitly discussed in [17] such as the modulation, multiplication, and modulation and translation. We also present a simple example to show the differences between the WVD-LCT and the WVD.

A. Definition of WVD-LCT

Based on the definition of the classical Wigner-Ville transform associated with the FT, we obtain a definition of the WVD-LCT by replacing the kernel of the FT with the kernel of the LCT in the classical WVT definition (8).

Definition 3. *For $f,\, g \in L^2(\mathbb{R})$, the cross WVD-LCT of f and g is defined by*

$$\mathcal{W}_{f,g}^A(t,\omega) = \int_{\mathbb{R}} f(t+\frac{x}{2})\overline{g(t-\frac{x}{2})}\,\frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2)}\,dx. \quad (9)$$

Suppose the kernel of the LCT with matrix parameter A is given by

$$K_A(x,\omega) = \frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2)}, \quad (10)$$

then (9) takes the form

$$\mathcal{W}_{f,g}^A(t,\omega) = \int_{\mathbb{R}} R_{f,g}(t,x)K_A(x,\omega)\,dx, \quad (11)$$

where $R_{f,g}(t,x) = f(t+\frac{x}{2})\overline{g(t-\frac{x}{2})}$. Notice that if $f=g$, then $W_{f,f}^A(t,\omega) = W_f^A(t,\omega)$ is called the auto WVD-LCT. That is,

$$\mathcal{W}_{f}^A(t,\omega) = \int_{\mathbb{R}} f(t+\frac{x}{2})\overline{f(t-\frac{x}{2})}\,\frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2)}\,dx. \quad (12)$$

Often both the cross WVD-LCT and auto WVD-LCT are usually referred to simply as the WVD-LCT. It directly follows from Definition 3 that the WVD-LCT is the LCT of the function $R_{f,g}(t,x)$ with respect to x. In other words,

$$\mathcal{W}_{f,g}^A(t,\omega) = L_A\{R_{f,g}(t,x)\}(\omega). \quad (13)$$

Note that for $f,\, g \in L^2(\mathbb{R})$, the WVD-LCT satisfies

$$|\mathcal{W}_{f,g}^A(t,\omega)|^2 \le \frac{2}{\pi|b|}\|f\|^2\|g\|^2. \quad (14)$$

For convenience, we consider an example of the WVD-LCT.

Example 1. *Let us find the WVD-LCT of a Gaussian signal*

$$f(t) = (\pi\sigma^2)^{-1/4}e^{-t^2/2\sigma^2}.$$

Using the LCT of the Gaussian function (see [5]), it is not difficult to get

$$\mathcal{W}_{f}^A(t,\omega) = \frac{e^{-t^2/\sigma^2}}{\pi\sqrt{2\sigma^2 ai - b^2}}\,e^{\frac{\omega^2}{2}\left(\frac{(2c\sigma^2-4d\sigma^2)+(bd+8c\sigma^2)}{b^2+4a^2\sigma^4}\right)}.$$

We can obtain the modulus of the WVD-LCT of f as

$$|\mathcal{W}_{f}^A(t,\omega)| = \frac{e^{-t^2/\sigma^2}}{\pi\sqrt{4\sigma^4a^2+b^4}}\,e^{\frac{\omega^2}{2}\left(\frac{2c\sigma^2-4d\sigma^2}{b^2+4a^2\sigma^4}\right)}.$$

B. General Properties of WVD-LCT

Here we present some general properties of the WVD-LCT and their detailed proofs. We shall see the differences between the classical WVT and WVD-LCT.

Theorem 1 (Nonlinearity)**.** *Suppose that we express a signal as the sum of two pieces: $h \in L^2(\mathbb{R}), h(t) = f(t)+g(t)$, then we get*

$$\mathcal{W}_{h}^A(t,\omega) = \mathcal{W}_{f}^A(t,\omega)+\mathcal{W}_{g}^A(t,\omega)+\mathcal{W}_{f,g}^A(t,\omega)+\mathcal{W}_{g,f}^A(t,\omega).$$

This property indicates that the WVD-LCT does not satisfy the superposition principle, which is not suitable to the analysis of multicomponent signals. Notice that if $A = (0,1;-1,0)$, above can be rewritten as

$$\mathcal{W}_{h}^A(t,\omega) = \mathcal{W}_{f}^A(t,\omega)+\mathcal{W}_{g}^A(t,\omega)+2\mathrm{Re}\,\mathcal{W}_{f,g}^A(t,\omega).$$

Theorem 2 (Complex conjugation)**.** *For $f,\, g \in L^2(\mathbb{R})$, we have*

$$\overline{\mathcal{W}_{f,g}^A(t,\omega)} = \mathcal{W}_{g,f}^{A^{-1}}(t,\omega). \quad (15)$$

Proof: Applying the definition of the WVD-LCT and inverse of the matrix parameter $A=(a,b;c,d)$, it is easy to obtain

$$\begin{aligned}&\overline{\mathcal{W}_{f,g}^A(t,\omega)}\\ &= \overline{\int_{\mathbb{R}} f(t+\frac{x}{2})\overline{g(t-\frac{x}{2})}\,\frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2)}\,dx}\\ &= \frac{1}{\sqrt{-2\pi bi}}\int_{\mathbb{R}} g(t-\frac{x}{2})\overline{f(t+\frac{x}{2})}\,e^{-i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2)}\,dx\\ &= \mathcal{W}_{g,f}^{A^{-1}}(t,\omega),\end{aligned}$$

which proves the theorem. ∎

For $A=(0,1;-1,0)$, (15) becomes $\overline{\mathcal{W}_{f,g}^A(t,\omega)} = \mathcal{W}_{g,f}^A(t,\omega)$, which resembles the complex conjugation property of the classical WVT.

We obtain two following properties, which are similar to the classical WVT properties.

Theorem 3 (Multiplication)**.** *Let $f,\, g \in L^2(\mathbb{R})$ and $Mf(t) = tf(t)$. Then*

$$2t\mathcal{W}_{f,g}^A(t,\omega) = \mathcal{W}_{Mf,g}^A(t,\omega)+\mathcal{W}_{f,Mg}^A(t,\omega).$$

Proof: A straightforward computation shows that

$$\begin{aligned}&2t\mathcal{W}_{f,g}^A(t,\omega)\\ &= \int_{\mathbb{R}}\left\{(t+\frac{x}{2})f(t+\frac{x}{2})\overline{g(t-\frac{x}{2})}+f(t+\frac{x}{2})\overline{g(t-\frac{x}{2})}\right\}\\ &\quad\times\frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2)}\,dx\\ &= \mathcal{W}_{Mf,g}^A(t,\omega)+\mathcal{W}_{f,Mg}^A(t,\omega),\end{aligned}$$

which completes the proof. ∎

Theorem 4 (Translation)**.** *Let $f,\, g \in L^2(\mathbb{R})$ be two signals. If the signals are translated in the spatial domain, its WVD-LCT is also translated, that is,*

$$\mathcal{W}_{\tau_k f,\tau_k g}^A(t,\omega) = \mathcal{W}_{f,g}^A(t-k,\omega). \quad (16)$$

In the following theorems we shall see the differences between the WVD-LCT properties and the classical WVT properties. These results are easily proved by applying of the LCT properties.

Theorem 5 (Modulation)**.** *For $f,\ g \in L^2(\mathbb{R})$, we have*

$$\mathcal{W}^A_{M_{\omega_0}f,M_{\omega_0}g}(t,\omega)=e^{id\omega\omega_0}e^{-i\frac{1}{2}db\omega_0^2}\mathcal{W}^A_{f,g}(t,\omega-\omega_0 b). \quad (17)$$

Proof: Direct computations show that

$$\begin{aligned}&\mathcal{W}^A_{M_{\omega_0}f,M_{\omega_0}g}(t,\omega)\\&=\int_{\mathbb{R}}e^{i\omega_0 x}f(t+\frac{x}{2})\overline{g(t-\frac{x}{2})}\,\frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x\omega+\frac{d}{b}\omega^2)}\,dx\\&=\int_{\mathbb{R}}f(t+\frac{x}{2})\overline{g(t-\frac{x}{2})}\,\frac{1}{\sqrt{2\pi bi}}\\&\quad\times e^{i\frac{1}{2}\left(\frac{a}{b}x^2-\frac{2}{b}x(\omega-\omega_0 b)+\frac{d}{b}((\omega-\omega_0 b)^2+2(\omega-\omega_0 b)\omega_0 b+\omega_0^2 b^2)\right)}\,dx.\end{aligned}$$

Applying Definition 3 finishes the proof. ∎

Theorem 6 (Modulation and Translation)**.** *For $f,\ g\in L^2(\mathbb{R})$, the following holds*

$$\begin{aligned}&\mathcal{W}^A_{M_{\omega_0}\tau_k f,M_{\omega_0}\tau_k g}(t,\omega)\\&\qquad=e^{id\omega\omega_0}e^{-i\frac{1}{2}db\omega_0^2}\mathcal{W}^A_{f,g}(t-k,\omega-\omega_0 b) \quad (18)\\&\mathcal{W}^A_{M_{\omega_0}f,\tau_k g}(t,\omega)\\&\qquad=e^{-i\omega_0 t}e^{id\omega\omega_0}e^{-i\frac{1}{2}db\omega_0^2}\mathcal{W}^A_{f,g}(t,\omega-\frac{1}{2}\omega_0 b) \quad (19)\\&\mathcal{W}^A_{\tau_k f,M_{\omega_0}\tau_k g}(t,\omega)\\&\qquad=e^{-i\omega_0 t}e^{id\omega\omega_0}e^{-i\frac{1}{2}db\omega_0^2}\mathcal{W}^A_{f,g}(t,\omega-\frac{1}{2}\omega_0 b). \quad (20)\end{aligned}$$

Proof: For (18), we have

$$\begin{aligned}&\mathcal{W}^A_{M_{\omega_0}\tau_k f,M_{\omega_0}\tau_k g}(t,\omega)\\&\quad=\int_{\mathbb{R}}f(t-k+\frac{x}{2})\overline{g(t-k-\frac{x}{2})}\\&\qquad\times\frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}(\frac{a}{b}x^2-\frac{2}{b}x(\omega-\omega_b)+\frac{d}{b}\omega^2)}\,dx.\end{aligned}$$

Using Theorem 4 and then applying the modulation property of the LCT, it is easy to verify that the above expression has the following form:

$$\mathcal{W}^A_{M_{\omega_0}\tau_k f,M_{\omega_0}\tau_k g}(t,\omega)=e^{id\omega\omega_0}e^{-i\frac{1}{2}db\omega_0^2}\mathcal{W}^A_{f,g}(t-k,\omega-\omega_0 b).$$

By similar arguments, we can prove equations (19) and (20). ∎

C. Main Properties of WVD-LCT

This subsection describes the main properties of the WVD-LCT. We now first establish a reconstruction formula.

Theorem 7 (Reconstruction formula)**.** *For $f,\ g\in L^2(\mathbb{R})$, we get the following inversion formula of the WVD-LCT:*

$$f(t)=\frac{1}{\sqrt{-2\pi bi}}\frac{1}{\overline{g(0)}}\int_{\mathbb{R}}\mathcal{W}^A_{f,g}(\frac{t}{2},\omega)e^{i\frac{1}{2}(\frac{-d}{b}\omega^2+\frac{2}{b}t\omega-\frac{a}{b}t^2)}\,d\omega. \quad (21)$$

Proof: By (13), we have

$$\begin{aligned}\mathcal{W}^A_{f,g}(t,\omega)&=L_A\{R_{f,g}(t,x)\}(\omega)\\&=L_A\left\{f(t+\frac{x}{2})\overline{g(t-\frac{x}{2})}\right\}(\omega).\end{aligned}$$

By (3), we obtain

$$\begin{aligned}&f(t+\frac{x}{2})\overline{g(t-\frac{x}{2})}\\&\quad=L_A^{-1}\{\mathcal{W}^A_{f,g}(t,\omega)\}\\&\quad=\frac{1}{\sqrt{-2\pi bi}}\int_{\mathbb{R}}\mathcal{W}^A_{f,g}(t,\omega)\,e^{i\frac{1}{2}(\frac{-d}{b}\omega^2+\frac{2}{b}x\omega-\frac{a}{b}x^2)}\,d\omega.\end{aligned}$$

With the change of variables $t_1=t+x/2$ and $t_2=t-x/2$, the above yields

$$\begin{aligned}f(t_1)\overline{g(t_2)}=&\frac{1}{\sqrt{-2\pi bi}}\int_{\mathbb{R}}\mathcal{W}^A_{f,g}\left(\frac{t_1+t_2}{2},\omega\right)\\&\times e^{i\frac{1}{2}(\frac{-d}{b}\omega^2+\frac{2}{b}(t_1-t_2)\omega-\frac{a}{b}(t_1-t_2)^2)}\,d\omega.\end{aligned}$$

By putting $t_2=0$, we obtain for any t_1

$$\begin{aligned}f(t_1)=&\frac{1}{\sqrt{-2\pi bi}}\frac{1}{\overline{g(0)}}\int_{\mathbb{R}}\mathcal{W}^A_{f,g}\left(\frac{t_1}{2},\omega\right)\\&\times e^{i\frac{1}{2}(\frac{-d}{b}\omega^2+\frac{2}{b}t_1\omega-\frac{a}{b}t_1^2)}\,d\omega,\end{aligned}$$

which is the desired reconstruction formula. ∎

Theorem 8 (Moyal's formula)**.** *For $f_1,\ f_2,\ g_1,\ g_2\in L^2(\mathbb{R})$, we have*

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{W}^A_{f_1,g_1}(t,\omega)\overline{\mathcal{W}^A_{f_2,g_2}(t,\omega)}\,d\omega dt=2(f_1,f_2)\overline{(g_1,g_2)}. \quad (22)$$

In particular, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}|\mathcal{W}^A_{f,g}(t,\omega)|^2\,d\omega dt=2\|f\|^2\|g\|^2, \quad (23)$$

and

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{W}^A_f(t,\omega)\overline{\mathcal{W}^A_g(t,\omega)}\,d\omega dt=2|(f,g)|^2. \quad (24)$$

Proof: By applying Parseval's formula to the integration with respect to ω in the left-hand side of (22), we have

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{W}^A_{f_1,g_1}(t,\omega)\overline{\mathcal{W}^A_{f_2,g_2}(t,\omega)}\,d\omega dt\\&\quad=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}L_A\{R_{f_1,g_1}(t,x)\}\overline{L_A\{R_{f_2,g_2}(t,x)\}}\,d\omega\right)dt\\&\quad=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}R_{f_1,g_1}(t,x)\overline{R_{f_2,g_2}(t,x)}\,d\omega\right)dt.\end{aligned}$$

Subsequent calculations reveal that

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{W}^A_{f_1,g_1}(t,\omega)\overline{\mathcal{W}^A_{f_2,g_2}(t,\omega)}\,d\omega dt\\&\quad=\int_{\mathbb{R}}\int_{\mathbb{R}}f_1(t+\frac{x}{2})\overline{g_1(t-\frac{x}{2})}g_2(t-\frac{x}{2})\overline{f_2(t-\frac{x}{2})}\,dxdt.\end{aligned}$$

With the change of variables $y = t + x/2$ and $z = t - x/2$, we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{W}^A_{f_1,g_1}(t,\omega)\overline{\mathcal{W}^A_{f_2,g_2}(t,\omega)}\,d\omega dt = 2\int_{\mathbb{R}}\int_{\mathbb{R}} f_1(y)\overline{g_1(z)}g_2(z)\overline{f_2(y)}\,dydz.$$

Interchanging the order of integrals, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{W}^A_{f_1,g_1}(t,\omega)\overline{\mathcal{W}^A_{f_2,g_2}(t,\omega)}\,d\omega dt = 2(f_1,f_2)\overline{(g_1,g_2)}.$$

Equations (23) and (24) are direct consequences of (22). Note that equation (24) states that the signal energy is preserved by the WVD-LCT. ■

IV. Convolution and Correlation for WVD-LCT

The convolution and correlation are fundamental signal processing algorithms in the theory of linear time-invariant (LTI) systems. In engineering, they have been widely used for various template matchings. In the following, we first define the convolution and correlation for the LCT (see, for example, [19], [20]). They are extensions of the convolution and correlation definitions from the FT to the LCT domain. We then establish the new convolution and correlation for the WVD-LCT.

Definition 4 (LCT Convolution). *Let $W(x,t)$ be a weight function. For $f,\, g \in L^2(\mathbb{R})$, we define the convolution operator of the LCT by*

$$(f * g)(t) = \int_{\mathbb{R}} f(x)g(t-x)W(x,t)\,dx. \tag{25}$$

Hereinafter, we assume that the weight function is $W(x,t) = e^{i\frac{a}{b}2x(x-t)}$. As a consequence of the above definition, we get the following important result.

Theorem 9 (WVD-LCT Convolution). *For $f,\, g \in L^2(\mathbb{R})$, the following result holds*

$$\mathcal{W}^A_{f*g}(t,\omega) = \sqrt{2\pi bi}\, e^{-i\frac{1}{2}\frac{d}{b}\omega^2} \times \int_{\mathbb{R}} \mathcal{W}^A_f(u,\omega)\mathcal{W}^A_g(t-u,\omega)\, e^{i\frac{a}{b}(4u(t-u))}\,du. \tag{26}$$

If the parameter of the WVD-LCT convolution changes $A = (0,1;-1,0)$, (26) reduces to convolution theorem in the classical WVT domain as follows

$$\mathcal{W}^A_{f*g}(t,\omega) = \sqrt{2\pi i}\int_{\mathbb{R}} \mathcal{W}^A_f(u,\omega)\mathcal{W}^A_g(t-u,\omega)\,du. \tag{27}$$

Proof: By (9) and (25), we immediately obtain

$$\begin{aligned}\mathcal{W}^A_{f*g}(t,\omega) &= \int_{\mathbb{R}} (f*g)(t+\frac{\eta}{2})(\overline{f}*\overline{g})(t-\frac{\eta}{2}) \times \frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}(\frac{a}{b}\eta^2-\frac{2}{b}\eta\omega+\frac{d}{b}\omega^2)}\,d\eta \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)g(t+\frac{\eta}{2}-x)e^{i\frac{a}{b}2x(x-(t+\frac{\eta}{2}))}\,dx \\ &\quad\times \int_{\mathbb{R}} \overline{f(y)g(t-\frac{\eta}{2}-y)e^{i\frac{a}{b}2y(y-(t-\frac{\eta}{2}))}}\,dy \\ &\quad\times \frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}(\frac{a}{b}\eta^2-\frac{2}{b}\eta\omega+\frac{d}{b}\omega^2)}\,d\eta.\end{aligned}$$

Putting $x = u + \frac{p}{2}$, $y = u - \frac{p}{2}$, and $\eta = p + q$, we have

$$\begin{aligned}&\mathcal{W}^A_{f*g}(t,\omega) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u+\frac{p}{2})g(t+\frac{\eta}{2}-(u+\frac{p}{2})) \\ &\quad\times e^{-i\frac{a}{b}2(u+\frac{p}{2})((t+\frac{\eta}{2})-(u+\frac{p}{2}))}\overline{f(u-\frac{p}{2})} \\ &\quad\times \overline{g\left((t-\frac{\eta}{2})-(u-\frac{p}{2})\right)}e^{-i\frac{a}{b}2(u-\frac{p}{2})((t-\frac{\eta}{2})-(u-\frac{p}{2}))} \\ &\quad\times \frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}(\frac{a}{b}\eta^2-\frac{2}{b}\eta\omega+\frac{d}{b}\omega^2)}\,dpdqdu \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(u+\frac{p}{2})\overline{f(u-\frac{p}{2})}\frac{1}{\sqrt{2\pi bi}}\, e^{i\frac{1}{2}\frac{a}{b}p^2}e^{-i\frac{p}{b}\omega}e^{i\frac{1}{2}\frac{d}{b}\omega^2}\,dp \\ &\quad\times \int_{\mathbb{R}} g(t-u+\frac{q}{2}))\overline{g(t-u-\frac{q}{2})}\, e^{i\frac{1}{2}\frac{a}{b}q^2}e^{-i\frac{q}{b}\omega}\,dq \\ &\quad\times e^{-i\frac{a}{b}(4u(t-u))}\,du.\end{aligned} \tag{28}$$

Multiplying both sides of (28) by $\frac{1}{\sqrt{2\pi bi}}\, e^{i\frac{1}{2}\frac{d}{b}\omega^2}$, we have

$$\frac{1}{\sqrt{2\pi bi}}\, e^{i\frac{1}{2}\frac{d}{b}\omega^2}\mathcal{W}^A_{f*g}(t,\omega) = \int_{\mathbb{R}} \mathcal{W}^A_f(u,\omega)\mathcal{W}^A_g(t-u,\omega)\, e^{i\frac{a}{b}(4u(t-u))}\,du,$$

which proves the theorem. ■

Next, we will derive the correlation theorem in the WVD-LCT domain. Let us define the correlation for the LCT.

Definition 5 (LCT Correlation). *For $f,\, g \in L^2(\mathbb{R})$, we define the correlation operator of the WVD-LCT as*

$$(f \circ g)(t) = \int_{\mathbb{R}} \overline{f(x)}g(x+t)e^{i\frac{a}{b}2x(x+t)}\,dx. \tag{29}$$

Then, we obtain the following important theorem.

Theorem 10 (WVD-LCT Correlation). *For $f,\, g \in L^2(\mathbb{R})$, the following result holds*

$$\mathcal{W}^A_{f\circ g}(t,\omega) = \sqrt{2\pi bi}\, e^{-i\frac{1}{2}\frac{d}{b}\omega^2} \times \int_{\mathbb{R}} \mathcal{W}^A_{\overline{f}}(u,-\omega)\mathcal{W}^A_g(t+u,\omega)\, e^{i\frac{a}{b}(4u(t+u))}\,du. \tag{30}$$

In particular, the case when $A = (0,1;-1,0)$, (30) will lead to the following form

$$\mathcal{W}^A_{f\circ g}(t,\omega) = \sqrt{2\pi i}\int_{\mathbb{R}} \mathcal{W}^A_{\overline{f}}(u,-\omega)\mathcal{W}^A_g(t+u,\omega)\,du. \tag{31}$$

Proof: Applying (9) to the left-hand side of (30), we have

$$\mathcal{W}^A_{f\circ g}(t,\omega) = \int_{\mathbb{R}} (f\circ g)(t+\frac{\eta}{2})(\overline{f}\circ\overline{g})(t-\frac{\eta}{2}) \times \frac{1}{\sqrt{2\pi bi}}e^{i\frac{1}{2}(\frac{a}{b}\eta^2-\frac{2}{b}\eta\omega+\frac{d}{b}\omega^2)}\,d\eta$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f(x)} g(x + (t + \frac{\eta}{2})) e^{i \frac{a}{b} 2x(x + (t + \frac{\eta}{2}))} \, dx$$
$$\times \int_{\mathbb{R}} \overline{g(y + (t - \frac{\eta}{2}))} f(y) e^{i \frac{a}{b} 2y(y + (t - \frac{\eta}{2}))} \, dy$$
$$\times \frac{1}{\sqrt{2\pi bi}} e^{i \frac{1}{2}(\frac{a}{b}\eta^2 - \frac{2}{b}\eta\omega + \frac{d}{b}\omega^2)} \, d\eta.$$

Putting $x = u + \frac{p}{2}$, $y = u - \frac{p}{2}$, and $\eta = q - p$, we obtain

$$\begin{aligned}
&\mathcal{W}^A_{f \circ g}(t, \omega) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f(u + \frac{p}{2})} g((u + \frac{p}{2}) + (u + \frac{\eta}{2})) \\
&\quad \times e^{i \frac{a}{b} 2(u + \frac{p}{2})((t + \frac{p}{2}) + (u + \frac{\eta}{2}))} f(u - \frac{p}{2}) \\
&\quad \times \overline{g\left((u - \frac{p}{2}) + (t - \frac{\eta}{2})\right)} e^{i \frac{a}{b} 2(u - \frac{p}{2})((t - \frac{\eta}{2}) + (u - \frac{p}{2}))} \\
&\quad \times \frac{1}{\sqrt{2\pi bi}} e^{i \frac{1}{2}(\frac{a}{b}\eta^2 - \frac{2}{b}\eta\omega + \frac{d}{b}\omega^2)} \, dpdqdu \qquad (32) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f(u + \frac{p}{2})} g\left((u + \frac{p}{2}) + (u + \frac{\eta}{2})\right) \\
&\quad \times e^{i \frac{a}{b} 2(u + \frac{p}{2})((t + u) + \frac{q}{2})} f(u - \frac{p}{2}) \overline{g\left((u - \frac{p}{2}) + (t - \frac{\eta}{2})\right)} \\
&\quad \times e^{i \frac{a}{b} 2(u - \frac{p}{2})((t + u) - \frac{q}{2})} \frac{1}{\sqrt{2\pi bi}} e^{i \frac{1}{2}(\frac{a}{b}\eta^2 - \frac{2}{b}\eta\omega + \frac{d}{b}\omega^2)} \, dpdqdu \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f(u + \frac{p}{2})} f(u - \frac{p}{2}) g(u + t + \frac{q}{2}) \overline{g(u + t - \frac{q}{2})} \\
&\quad \times e^{i \frac{a}{b} 2(u + \frac{p}{2})((t + u) + \frac{q}{2})} e^{i \frac{a}{b} 2(u - \frac{p}{2})((t + u) - \frac{q}{2})} \\
&\quad \times \frac{1}{\sqrt{2\pi bi}} e^{i \frac{1}{2}(\frac{a}{b}(q - p)^2 - \frac{2}{b}(q - p)\omega + \frac{d}{b}\omega^2)} \, dpdqdu \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f(u + \frac{p}{2})} f(u - \frac{p}{2}) \frac{1}{\sqrt{2\pi bi}} \, e^{i \frac{1}{2} \frac{a}{b} p^2} \\
&\quad \times e^{i \frac{p}{b}\omega} e^{i \frac{1}{2} \frac{d}{b}\omega^2} \, dp \\
&\quad \times \int_{\mathbb{R}} g(t + u + \frac{q}{2})) \overline{g(t + u - \frac{q}{2})} \\
&\quad \times e^{i \frac{1}{2} \frac{a}{b} q^2} e^{-i \frac{q}{b}\omega} \, dq \, e^{i \frac{a}{b}(4u(t + u))} \, du. \qquad (33)
\end{aligned}$$

Then, multiplying both sides of (32) by $\frac{1}{\sqrt{2\pi bi}} \, e^{i \frac{1}{2} \frac{d}{b}\omega^2}$, we have

$$\frac{1}{\sqrt{2\pi bi}} \, e^{i \frac{1}{2} \frac{d}{b}\omega^2} \mathcal{W}^A_{f \circ g}(t, \omega)$$
$$= \int_{\mathbb{R}} \mathcal{W}^A_f(u, -\omega) \mathcal{W}^A_g(t + u, \omega) \, e^{i \frac{a}{b}(4u(t + u))} \, du,$$

which completes the proof. ∎

V. Conclusion

The Wigner-Ville distribution associated with linear canonical transform (WVD-LCT) is a generalization of the WVD to the LCT domain. In this paper, we first have established the various properties of the WVD-LCT such as translation, modulation, multiplication, and Moyal's formula. Using properties of the WVD-LCT we have established the convolution and correlation of the WVD-LCT. The results are very important for their applications in digital signal and image processing.

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