Continuous quaternion fourier
and wavelet transforms

Mawardi Bahri
Department of Mathematics, Universitas Hasanuddin
Makassar 90245, Indonesia
mawardibahri@gmail.com

Ryuichi Ashino
Division of Mathematical Sciences, Osaka Kyoiku University
Osaka 582-8582, Japan
ashino9cc.osaka-kyoiku.ac.jp

Rémi Vaillancourt
Department of Mathematics and Statistics, University of Ottawa
Ottawa, ON, Canada K1N 6N5
remi@uottawa.ca

Received 4 June 2012
Revised 19 February 2013
Accepted 26 February 2013
Published 27 March 2014

A two-dimensional (2D) quaternion Fourier transform (QFT) defined with the kernel $e^{-\sqrt{3} \omega \cdot x}$ is proposed. Some fundamental properties, such as convolution, Plancherel and vector differential theorems, are established. The heat equation in quaternion algebra is presented as an example of the application of the QFT to partial differential equations. The wavelet transform is extended to quaternion algebra using the kernel of the QFT.

Keywords: Quaternion-valued function; quaternion algebra; quaternion Fourier transform.

AMS Subject Classification: 15A66, 42B10, 30G35

1. Introduction

It is well-known that every two-dimensional (2D) rotation around the origin in the plane $\mathbb{R}^2$ can be represented by the multiplication of the complex number $e^{i\theta} = \cos \theta + i \sin \theta$, $0 \leq \theta < 2\pi$. Similarly, every three-dimensional (3D) rotation in the space $\mathbb{R}^3$ can be represented by the multiplications of the quaternion $q$ from the left-hand side and its conjugate $\bar{q}$ from the right-hand side, where $q = \cos(\theta/2) + \alpha \sin(\theta/2)$ with a unit quaternion $\alpha$ representing the axis of rotation.
and angle $\theta$ of rotation, $0 \leq \theta < 2\pi$. For this reason, quaternions are commonly used in 3D computer graphics and computer vision. Recently, it has become popular to generalize the classical Fourier transform (FT) to quaternion algebra and used it in image analysis. For example, in order to compute the FT of a color image without separating it into three gray-scale images, a single and holistic FT, which treats a color image as a vector field, was proposed. In this framework, quaternions represent color image pixels. These extensions are broadly called the quaternion Fourier transform (QFT). Due to the noncommutative property of quaternion multiplication, there are at least three different types of 2D QFTs as follows (see Ell, Hitzer, Hitzer, Bahri, Hitzer, Hayashi and Ashino, Pei, Ding and Chan, Said, Bihan and Sangwine),

$$F_I q\{f\}(\omega) = \int_{\mathbb{R}^2} e^{-\mu_1 \omega \cdot x} f(x) d^2x, \quad \omega \cdot x = \omega_1 x_1 + \omega_2 x_2,$$

(1.1)

$$F_{II} q\{f\}(\omega) = \int_{\mathbb{R}^2} f(x) e^{-\mu_1 \omega \cdot x} d^2x, \quad \omega \cdot x = \omega_1 x_1 + \omega_2 x_2,$$

(1.2)

$$F_{III} q\{f\}(\omega) = \int_{\mathbb{R}^2} e^{-\mu_1 \omega_1 x_1} f(x) e^{-\mu_2 \omega_2 x_2} d^2x,$$

(1.3)

where $\mu_1$ and $\mu_2$ are any two unit pure quaternions ($\mu_1^2 = \mu_2^2 = -1$) that are orthogonal to each other. These three QFTs are so-called left-side, right-side and double-side, or type I, II and III, respectively.

Assefa et al. used QFTs of type II to establish the 2D quaternion Stockwell (QS) transform and then apply it for the analysis of local color image spectra. Recently, Guo and Zhu introduced the quaternion Fourier–Mellin moments as a generalization of traditional Fourier–Mellin moments to quaternion algebra. Several properties of this generalization are investigated using QFTs of type II.

On the other hand, a number of attempts have been previously made to generalize the classical wavelet transform (WT) to quaternion algebra. He, Zhao and Peng constructed the continuous quaternion wavelet transform (CQWT) of quaternion-valued functions. They also demonstrated a number of properties of these extended wavelets using the classical FT. Using the (two-sided) QFT, Traversoni proposed a discrete quaternion wavelet transform (QWT) which was applied by Bayro and Zhou, Xu and Yang. Olhede and Metikas have proposed the monogenic WT based on one-dimensional (1D) analytic WT. Recently, Bahri, Adjii and Zhao, and Bahri and Hitzer introduced an extension of the WT to Clifford algebra by means of the kernel of the Clifford FT. It was found that many WT properties still hold but others have to be modified.

In this paper, we concentrate on QFTs of type II with kernel $\mu_1 = \frac{i+j+k}{\sqrt{3}}$. We derive the shift, modulation, and convolution properties and also establish the Plancherel and vector differential theorems. Then, we define a CQWT using the kernel of QFTs of type II, which is essentially different from the kernel of QFTs of type III studied in Bahri, Ashino and Vaillancourt and Bahri.
2. Basics

Let us present the notation which will be used in this paper. The quaternion algebra over \( \mathbb{R} \), denoted by \( \mathbb{H} \), is an associative noncommutative four-dimensional (4D) algebra,

\[
\mathbb{H} = \{q = q_0 + iq_1 + jq_2 + kq_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\},
\]

which obeys Hamilton’s multiplication rules:

\[
ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1.
\]

For a quaternion \( q = q_0 + iq_1 + jq_2 + kq_3 \in \mathbb{H} \), \( q_0 \) is called the scalar (or real) part of \( q \), denoted by \( S_c(q) \), and \( iq_1 + jq_2 + kq_3 \) is called the vector (or pure) part of \( q \).

The quaternion conjugate of a quaternion \( q \) is given by

\[
\bar{q} = q_0 - iq_1 - jq_2 - kq_3, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}
\]

and it is an anti-involution, that is,

\[
qp = \bar{p}q.
\]

Using the conjugate (2.4) and the modulus of \( q \), we can define the reciprocal of \( q \in \mathbb{H}\backslash\{0\} \) by

\[
q^{-1} = \frac{\bar{q}}{|q|^2},
\]

which shows that \( \mathbb{H} \) is a normed division algebra. Any quaternion \( q \) can be written as

\[
q = |q|e^{\frac{i\theta + kj}{\sqrt{3}}}.
\]

where \( \theta = \arctan \frac{|S_c(q)|}{\text{Vec}(q)} \), \( 0 \leq \theta \leq \pi \), is the eigenangle or phase of \( q \). When \( |q| = 1 \), \( q \) is a unit quaternion. Euler’s and De Moivre’s formulas still hold in
quaternion space, that is, for a pure unit quaternion the following holds:

\[
e^{\frac{i+j+k}{\sqrt{3}} \theta} = \cos \theta + \frac{i+j+k}{\sqrt{3}} \sin \theta,
\]

\[
e^{\frac{i+j+k}{\sqrt{3}} n\theta} = \left( \cos \theta + \frac{i+j+k}{\sqrt{3}} \sin \theta \right)^n = \cos n\theta + \frac{i+j+k}{\sqrt{3}} \sin n\theta.
\]

As in the algebra of complex numbers, we can define three nontrivial algebra involutions for quaternions:

\[
\begin{align*}
\alpha(q) &= -iqi = -i(q_0 + iq_1 + jq_2 + kq_3)i = q_0 + iq_1 - jq_2 - kq_3, \\
\beta(q) &= -jqj = -j(q_0 + iq_1 + jq_2 + kq_3)j = q_0 - iq_1 + jq_2 - kq_3, \\
\gamma(q) &= -kqk = -k(q_0 + iq_1 + jq_2 + kq_3)k = q_0 - iq_1 - jq_2 + kq_3.
\end{align*}
\] (2.10)

It is convenient to introduce the inner product of two quaternion-valued functions, \(f, g : \mathbb{R}^2 \rightarrow \mathbb{H}\), as follows:

\[
(f, g)_{L^2(\mathbb{R}^2; \mathbb{H})} = \int_{\mathbb{R}^2} f(x) \overline{g(x)} \, d^2x.
\] (2.11)

In particular, if \(f = g\), then we obtain the associated norm:

\[
\|f\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2 = \int_{\mathbb{R}^2} |f(x)|^2 \, d^2x.
\] (2.12)

3. Basic Properties of QFT

Before giving the fundamental properties of 2D QFTs, we list their basic properties with detailed proofs. Most of them are essentially straightforward extensions of the 2D FT properties. Denote by \(\{e_1, e_2\}\) the standard basis of \(\mathbb{R}^2\).

**Definition 3.1.** Let \(f \in L^1(\mathbb{R}^2; \mathbb{H})\). The QFT of \(f\) is the function \(\mathcal{F}_q\{f\} : \mathbb{R}^2 \rightarrow \mathbb{H}\) defined by

\[
\mathcal{F}_q\{f\}(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^2} f(x) e^{-\frac{i+x}{\sqrt{3}} \omega \cdot x} \, d^2x,
\] (3.1)

where \(x = x_1e_1 + x_2e_2\), \(\omega = \omega_1e_1 + \omega_2e_2\) and \(e^{-\frac{i+x}{\sqrt{3}} \omega \cdot x}\) is called the quaternion Fourier kernel.

**Theorem 3.1.** Suppose that \(f \in L^2(\mathbb{R}^2; \mathbb{H})\) and \(\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2; \mathbb{H})\). Then the QFT is invertible with inverse

\[
\mathcal{F}^{-1}_q[\mathcal{F}_q\{f\}](x) = f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\omega) e^{\frac{i+x}{\sqrt{3}} \omega \cdot x} \, d^2\omega.
\] (3.2)

3.1. Left linearity

It is easy to show the following lemma.
Lemma 3.1. Let $f_1, f_2 \in L^2(\mathbb{R}^2; \mathbb{H})$. The QFT is left linear, that is,
\[ \mathcal{F}_q(\alpha f_1 + \beta f_2)(\omega) = \alpha \mathcal{F}_q(f_1)(\omega) + \beta \mathcal{F}_q(f_2)(\omega), \quad \alpha, \beta \in \mathbb{H}. \] (3.3)

Note that right linearity is not valid for the QFT.

3.2. Shift property

Lemma 3.2. The QFT of a shifted function is given by
\[ \mathcal{F}_q(f(x - b))(\omega) = \mathcal{F}_q(f)(\omega)e^{-\frac{i\omega}{\sqrt{2}}x.b}. \] (3.4)

Proof. Equation (3.1) gives
\[ \mathcal{F}_q(f(x - b))(\omega) = \int_{\mathbb{R}^2} f(x - b)e^{-\frac{i\omega}{\sqrt{2}}x}d^2x. \]
Substituting $t = x - b$ in the above expression, we have $x = t + b$ and $d^2x = d^2t$. Hence,
\[ \mathcal{F}_q(f(x - b))(\omega) = \int_{\mathbb{R}^2} f(t)e^{-\frac{i\omega}{\sqrt{2}}(t+b)}d^2t \]
\[ = \int_{\mathbb{R}^2} f(t)e^{-\frac{i\omega}{\sqrt{2}}t}d^2te^{-\frac{i\omega}{\sqrt{2}}b} \]
\[ = \mathcal{F}_q(f)(\omega)e^{-\frac{i\omega}{\sqrt{2}}b}. \]
This proves (3.4).

3.3. Scaling property

Lemma 3.3. Let $a \in \mathbb{R}\setminus\{0\}$. The QFT of the scaled function $f_a(x) = f(ax)$ is given by
\[ \mathcal{F}_q(f_a)(\omega) = \frac{1}{|a|^2} \mathcal{F}_q(f)\left(\frac{\omega}{a}\right). \] (3.5)

Proof. We first assume that $a > 0$. Definition 3.1 gives
\[ \mathcal{F}_q(f_a)(\omega) = \int_{\mathbb{R}^2} f(ax)e^{-\frac{i\omega}{\sqrt{2}}x}d^2x. \]
Substituting $u$ for $ax$, we have
\[ \mathcal{F}_q(f_a)(\omega) = \frac{1}{a^2} \int_{\mathbb{R}^2} f(u)e^{-\frac{i\omega}{\sqrt{2}}u}d^2u \]
\[ = \frac{1}{a^2} \mathcal{F}_q(f)\left(\frac{\omega}{a}\right). \]
For $a < 0$, we have
\[ \mathcal{F}_q(f_a)(\omega) = \frac{1}{(-a)^2} \mathcal{F}_q(f)\left(\frac{\omega}{a}\right), \]
which completes the proof.
3.4. Modulation property

Lemma 3.4. Let $\omega_0 \in \mathbb{R}^2$ and $F_0(x) = f(x)e^{\frac{1+i+j+k}{\sqrt{3}}\omega_0 \cdot x}$. Then, we have

$$\mathcal{F}_q\{F_0\}(\omega) = \mathcal{F}_q\{f\}(\omega - \omega_0).$$

(3.6)

Proof. Using Definition 3.1 and simplifying it, we obtain

$$\mathcal{F}_q\{F_0\}(\omega) = \int_{\mathbb{R}^2} f(x)e^{\frac{1+i+j+k}{\sqrt{3}}\omega_0 \cdot x}e^{-\frac{1+i+j+k}{\sqrt{3}}\omega \cdot x} d^2x$$

$$= \int_{\mathbb{R}^2} f(x)e^{-\frac{1+i+j+k}{\sqrt{3}}(\omega - \omega_0) \cdot x} d^2x$$

$$= \mathcal{F}_q\{f\}(\omega - \omega_0),$$

which was to be proved.

Notation 3.1. We use the conventional notation $f(x) \in \mathbb{R}$, which means that the function $f(x)$ is real-valued.

Remark 3.1. Note that this property is different from the usual modulation property of the 2D FT. If the modulation term is multiplied from the left, that is,

$$F_0(x) = e^{\frac{1+i+j+k}{\sqrt{3}}\omega_0 \cdot x}f(x),$$

then the modulation property holds only for

$$f(x) = f_0(x) + (i + j + k)f_1(x), \quad f_0(x), f_1(x) \in \mathbb{R}.$$

4. Main Properties of the QFT

This section describes important properties of the QFT, such as the Plancherel and convolution theorems. First, we establish the Plancherel theorem.

Theorem 4.1 (QFT Plancherel). Let $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$. Then, we have

$$(f, g)_{L^2(\mathbb{R}^2; \mathbb{H})} = \frac{1}{(2\pi)^2}(\mathcal{F}_q\{f\}, \mathcal{F}_q\{g\})_{L^2(\mathbb{R}^2; \mathbb{H})}.$$  \hspace{1cm} (4.1)

In particular, with $f = g$, we have the Parseval theorem:

$$\|f\|_{L^2(\mathbb{R}^2; \mathbb{H})} = \frac{1}{2\pi}\|\mathcal{F}_q\{f\}\|_{L^2(\mathbb{R}^2; \mathbb{H})}.  \hspace{1cm} (4.2)$$

Proof. Equation (4.1) follows from

$$\begin{align*}
(f, g)_{L^2(\mathbb{R}^2; \mathbb{H})} & = \int_{\mathbb{R}^2} f(x)g(x) d^2x \\
& = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\omega)e^{\frac{1+i+j+k}{\sqrt{3}}\omega \cdot x} d^2\omega \right] g(x) d^2x \\
& = \mathcal{F}_q\{f\}, \mathcal{F}_q\{g\})_{L^2(\mathbb{R}^2; \mathbb{H})}.
\end{align*}$$
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\[
\begin{align*}
(2.5) & \quad \frac{1}{(2\pi)^2} \int_{\mathbb{H}^2} F_q\{f\} (\omega) \left[ \int_{\mathbb{R}^2} g(x) e^{-\frac{i|x-y|}{\sqrt{2}} \omega \cdot x} d^2x \right] d^2\omega \\
& = \frac{1}{(2\pi)^2} \int_{\mathbb{H}^2} F_q\{f\} (\omega) F_q\{g\} (\omega) d^2\omega, \\
(4.3) & \\
\end{align*}
\]
which completes the proof of Theorem 4.1.

The most important property of the QFT for applications in signal processing is the convolution theorem. Due to the noncommutativity of quaternion multiplication, we have the following definition.

**Definition 4.1.** Let \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \). The convolution \( f \ast g \) is defined by

\[
(f \ast g)(x) = \int_{\mathbb{R}^2} f(y)g(x - y) d^2y. \tag{4.4}
\]

**Remark 4.1.** In general, \( f \ast g \neq g \ast f \) because the quaternion multiplication is not commutative:

\( f(y)g(x - y) \neq g(x - y)f(y) \).

**Theorem 4.2.** Let \( f, g \in L^2(\mathbb{R}^2; \mathbb{H}) \) be two quaternion-valued functions. If \( f \) has the following representation

\[
f(x) = f_0(x) + i f_1(x) + j f_2(x) + k f_3(x),
\]

then, the QFT of the convolution \( f \ast g \) is given by

\[
F_q\{f \ast g\} (\omega) = F_q\{g\} (\omega) F_q\{f_0\} (\omega) + i F_q\{g\} (\omega) F_q\{f_1\} (\omega)
\]
\[
\quad + j F_q\{g\} (\omega) F_q\{f_2\} (\omega) + k F_q\{g\} (\omega) F_q\{f_3\} (\omega)
\]
\[
\quad = F_q\{g\} (\omega) F_q\{f_0\} (\omega) + \alpha(F_q\{g\} (\omega)) i F_q\{f_1\} (\omega)
\]
\[
\quad + \beta(F_q\{g\} (\omega)) j F_q\{f_2\} (\omega) + \gamma(F_q\{g\} (\omega)) k F_q\{f_3\} (\omega), \tag{4.5}
\]
where \( \alpha, \beta \) and \( \gamma \) denote the three nontrivial automorphisms of the quaternion algebra defined in (2.10).

**Proof.** Applying the definition of the QFT on the left-hand side of (4.4), we have

\[
F_q\{f \ast g\} (\omega) = \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}^2} f(y)g(x - y) d^2y \right] e^{-\frac{i|x-y|}{\sqrt{2}} \omega \cdot x} d^2x
\]
\[
= \int_{\mathbb{R}^2} f(y) \left[ \int_{\mathbb{R}^2} g(x - y) e^{-\frac{i|x-y|}{\sqrt{2}} \omega \cdot x} d^2x \right] d^2y. \tag{4.6}
\]
Putting \( z = x - y \) and applying again the definition of the QFT, we can rewrite (4.6) as

\[
\mathcal{F}_q\{f \star g\}(\omega) = \int_{\mathbb{R}^2} f(y) \left[ \int_{\mathbb{R}^2} g(z) e^{-\frac{i+y^j+k^j}{\sqrt{3}} \omega (y+z)} \, dz \right] \, d^2 y
\]

\[
= \int_{\mathbb{R}^2} f(y) \left[ \int_{\mathbb{R}^2} g(z) e^{-\frac{i+y^j+k^j}{\sqrt{3}} \omega z} e^{-\frac{i+y^j+k^j}{\sqrt{3}} \omega y} \, dz \right] \, d^2 y
\]

\[
= \int_{\mathbb{R}^2} (f_0(y) + i f_1(y) + j f_2(y) + k f_3(y))[F_q\{g\}(\omega)] e^{-\frac{i+y^j+k^j}{\sqrt{3}} \omega y} \, d^2 y
\]

\[
= \int_{\mathbb{R}^2} (F_q\{g\}(\omega)f_0(y) + i F_q\{g\}(\omega)f_1(y) + j F_q\{g\}(\omega)f_2(y) + k F_q\{g\}(\omega)f_3(y)) e^{-\frac{i+y^j+k^j}{\sqrt{3}} \omega y} \, d^2 y
\]

\[
= F_q\{g\}(\omega) F_q\{f_0\}(\omega) + i F_q\{g\}(\omega) F_q\{f_1\}(\omega) + j F_q\{g\}(\omega) F_q\{f_2\}(\omega) + k F_q\{g\}(\omega) F_q\{f_3\}(\omega),
\]

which finishes the proof by (2.10).

As a special case of Theorem 4.2, we have the following lemma.

**Lemma 4.1.** (i) If \( F_q\{g\}(\omega) \in \mathbb{R} \), then

\[
\mathcal{F}_q\{f \star g\}(\omega) = F_q\{f\}(\omega) F_q\{g\}(\omega).
\]

(ii) If \( f(x) \in \mathbb{R} \), then

\[
\mathcal{F}_q\{f \star g\}(\omega) = F_q\{g\}(\omega) F_q\{f\}(\omega).
\]

Note that the convolution of two Gaussian functions is again a Gaussian function by Lemma 4.1.

Applying the inverse QFT to the left-hand side of (4.8), we have the following corollary, which is important for solving partial differential equations in quaternion algebra.

**Corollary 4.1.** Assume that \( F_q\{g\}(\omega) \in \mathbb{R} \). Then, we have

\[
\mathcal{F}_q^{-1}[F_q\{f\}F_q\{g\}](x) = (f \star g)(x).
\]

**Proof.** By the QFT inversion, we have

\[
\mathcal{F}_q^{-1}[F_q\{f\}F_q\{g\}](x) \overset{(3.2)}{=} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(y) e^{-\frac{i+y^j+k^j}{\sqrt{3}} \omega y} d^2 y F_q\{g\}(\omega)
\]

\[
\times e^{-\frac{i+y^j+k^j}{\sqrt{3}} \omega x} \, d^2 \omega
\]
In the second equality of (4.11), we used the assumption to interchange and combine the quaternion Fourier kernel. This completes the proof of (4.10).

5. Differentiation of QFT

Differentiation properties of 2D FT can be generalized to the proposed QFT without changing their invariant expressions (independence of coordinates) in terms of vector differentials and vector derivatives. First, we define the properties of vector differentials (compare to Hitzer and Bahri,\textsuperscript{18} Bahri and Hitzer\textsuperscript{5}).

Let $(\mathbb{R}^n, Q) = V$ be a real vector space with basis $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ and non-degenerate quadratic form $Q : V \rightarrow \mathbb{R}$:

$$Q(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2, \quad p + q = n, \ p, q \in \mathbb{Z}_+,$$

where $x = \sum_{i=1}^n x_i \tilde{e}_i$ represents an arbitrary element of $(\mathbb{R}^n, Q)$ and $\mathbb{Z}_+ = \{n \in \mathbb{Z} | n > 0\}$. It is required that the following basic multiplication rules hold:

$$\tilde{e}_i^2 = 1, \quad 1 \leq i \leq p,$$
$$\tilde{e}_i^2 = -1, \quad p + 1 \leq i \leq n,$$
$$\tilde{e}_i \tilde{e}_j + \tilde{e}_j \tilde{e}_i = 0, \quad i \neq j, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n.$$

Sylvester’s theorem guarantees that $p$ and $q$ do not depend on the choice of the basis $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$. We use the abbreviation $\mathbb{R}^{p,q}$ for $(\mathbb{R}^n, Q)$. The pair $p, q$ is called the signature of $\mathbb{R}^{p,q}$. The spaces $\mathbb{R}^{n,0}$ are called Euclidean spaces while all spaces of type $\mathbb{R}^{0,n}$ are called anti-Euclidean spaces.

**Definition 5.1.** Let $\mathbf{a} \in \mathbb{R}^{0,2}$. The vector differential $\mathbf{a} \cdot \nabla$ along the direction $\mathbf{a}$ is defined by

$$\mathbf{a} \cdot \nabla = a_1 \partial_1 + a_2 \partial_2, \quad (5.1)$$

where $\nabla = \tilde{e}_1 \partial_1 + \tilde{e}_2 \partial_2$, $a_k = \tilde{e}_k$ and $\partial_k = \frac{\partial}{\partial x_k}$, $k = 1, 2$.

Applying the vector derivative $\nabla$ twice, we have

$$\nabla^2 = \nabla \nabla = \left( \tilde{e}_1 \frac{\partial}{\partial x_1} + \tilde{e}_2 \frac{\partial}{\partial x_2} \right) \left( \tilde{e}_1 \frac{\partial}{\partial x_1} + \tilde{e}_2 \frac{\partial}{\partial x_2} \right) = - \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right), \quad (5.2)$$

where we used the fact that $\tilde{e}_1^2 = \tilde{e}_2^2 = -1$ and $\tilde{e}_1 \tilde{e}_2 = -\tilde{e}_2 \tilde{e}_1$. 

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Theorem 5.1 (Vector differential). The QFT of the vector differential of \( f \in L^2(\mathbb{R}^{0,2}; \mathbb{H}) \) is given by

\[
\mathcal{F}_q \{ a \cdot \nabla f \}(\omega) = \mathcal{F}_q \{ f \}(\omega) \frac{i + j + k}{\sqrt{3}} a \cdot \omega. \tag{5.3}
\]

Proof. Applying the vector differential \( a \cdot \nabla \) to the inversion formula (3.2), we have

\[
a \cdot \nabla f(x) = a \cdot \nabla \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q \{ f \}(\omega) e^{i \frac{i + j + k}{\sqrt{3}} \omega \cdot x} d^2 \omega
\]

\[
= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q \{ f \}(\omega) (a \cdot \nabla e^{i \frac{i + j + k}{\sqrt{3}} \omega \cdot x}) d^2 \omega
\]

\[
= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q \{ f \}(\omega) \left( \frac{i + j + k}{\sqrt{3}} a \cdot \omega \right) e^{i \frac{i + j + k}{\sqrt{3}} \omega \cdot x} d^2 \omega
\]

\[
= \mathcal{F}_q^{-1} \left[ \mathcal{F}_q \{ f \}(\omega) \frac{i + j + k}{\sqrt{3}} a \cdot \omega \right] (x),
\]

which implies (5.3).

Example 5.1. Setting \( a = \tilde{e}_k, \; k = 1, 2 \), we have the QFT of partial derivatives of \( f(x) \),

\[
\mathcal{F}_q \{ \partial_k f \}(\omega) = \mathcal{F}_q \{ f \}(\omega) \frac{i + j + k}{\sqrt{3}} \omega_k, \; k = 1, 2. \tag{5.4}
\]

A similar calculation gives the following theorem.

Theorem 5.2. Let \( a, b \in \mathbb{R}^{0,2} \). Then, we have

\[
\mathcal{F}_q \{(a \cdot \nabla)(b \cdot \nabla)f \}(\omega) = -a \cdot \omega b \cdot \omega \mathcal{F}_q \{ f \}(\omega). \tag{5.5}
\]

Example 5.2. For \( a = \tilde{e}_k, \; b = \tilde{e}_l \), we have

\[
\mathcal{F}_q \{ \partial_k \partial_l \}(\omega) = -\omega_k \omega_l \mathcal{F}_q \{ f \}(\omega), \; k, l = 1, 2. \tag{5.6}
\]

Theorem 5.3. Let \( x \in \mathbb{R}^{0,2} \). Then, the QFT of the mth vector moment is given by

\[
\mathcal{F}_q \{ x^m f(x) \}(\omega) = \nabla^m_\omega \mathcal{F}_q \{ f \}(\omega) \left( \frac{i + j + k}{\sqrt{3}} \right)^m, \; m \in \mathbb{N}, \tag{5.7}
\]

where \( \nabla_\omega \) is the vector derivative with respect to the vector variable index \( \omega \), i.e. \( \nabla_\omega = \tilde{e}_1 \partial_{\omega_1} + \tilde{e}_2 \partial_{\omega_2} \).

Proof. A direct calculation of the first vector moment \( (m = 1) \) gives

\[
\mathcal{F}_q \{ x f(x) \}(\omega) = \int_{\mathbb{R}^2} x f(x) e^{-i \frac{i + j + k}{\sqrt{3}} \omega \cdot x} d^2 x
\]

\[
= \int_{\mathbb{R}^2} \nabla_\omega f(x) \frac{i + j + k}{\sqrt{3}} e^{-i \frac{i + j + k}{\sqrt{3}} \omega \cdot x} d^2 x
\]
Therefore, we have

\[ \nabla \omega f(x) = \nabla \omega \int_{\mathbb{R}^2} f(x) \frac{i + j + k}{\sqrt{3}} e^{-\frac{i + j + k}{\sqrt{3}} \omega \cdot x} d^2x \]

\[ = \nabla \omega \int_{\mathbb{R}^2} f(x) e^{-\frac{i + j + k}{\sqrt{3}} \omega \cdot x} d^2x i + j + k \]

\[ = \nabla \omega \mathcal{F}_q \{ f \} (\omega) \frac{i + j + k}{\sqrt{3}}. \]

In the fourth equality, we used the fact that

\[ e^{-\frac{i + j + k}{\sqrt{3}} \omega \cdot x} i + j + k e^{-\frac{i + j + k}{\sqrt{3}} \omega \cdot x}. \quad (5.8) \]

Repeating this procedure \( m - 1 \) times, we have

\[ \mathcal{F}_q \{ x^m f(x) \} (\omega) = \nabla \omega \mathcal{F}_q \{ f \} (\omega) \left( \frac{i + j + k}{\sqrt{3}} \right)^m, \quad (5.9) \]

which completes the proof.

**Theorem 5.4.** The QFT of the \( m \)th vector derivative is given by

\[ \mathcal{F}_q \{ \nabla^m f \} (\omega) = \mathcal{F}_q \{ f \} (\omega) \left( \frac{i + j + k}{\sqrt{3}} \right)^m, \quad m \in \mathbb{N}. \quad (5.10) \]

In particular, the case of the Laplacian \( \Delta = \nabla^2 \) is

\[ \mathcal{F}_q \{ \Delta f \} (\omega) = -\omega^2 \mathcal{F}_q \{ f \} (\omega). \quad (5.11) \]

**Proof.** A simple computation gives

\[ \nabla f(x) = \nabla \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q \{ f \} (\omega) e^{\frac{i + j + k}{\sqrt{3}} \omega \cdot x} d^2\omega \]

\[ = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q \{ f \} (\omega) \frac{i + j + k}{\sqrt{3}} \omega e^{\frac{i + j + k}{\sqrt{3}} \omega \cdot x} d^2\omega \]

\[ = \mathcal{F}_q^{-1} \left[ \mathcal{F}_q \{ f \} (\omega) \frac{i + j + k}{\sqrt{3}} \omega \right] (x). \quad (5.12) \]

Therefore, we have

\[ \mathcal{F}_q \{ \nabla f \} (\omega) = \mathcal{F}_q \{ f \} (\omega) \frac{i + j + k}{\sqrt{3}} \omega. \]

Applying the vector differential, \( \nabla \), to Eq. (5.12) once more, we have

\[ \mathcal{F}_q \{ \nabla^2 f \} = \mathcal{F}_q \{ \nabla (\nabla f) \} \]

\[ = \mathcal{F}_q \{ \nabla f \} (\omega) \frac{i + j + k}{\sqrt{3}} \omega \]

\[ = \mathcal{F}_q \{ f \} (\omega) \left( \frac{i + j + k}{\sqrt{3}} \omega \right)^2 \]

\[ = -\omega^2 \mathcal{F}_q \{ f \} (\omega). \quad (5.13) \]
Then, mathematical induction implies
\[ \mathcal{F}_q \{ \nabla^m f \} = \mathcal{F}_q \{ f \} \left( \frac{i + j + k}{\sqrt{3}} \omega \right)^m, \quad m \in \mathbb{N}. \] (5.14)

**Remark 5.1.** Notice that \( \omega^2 = \omega \cdot \omega + \omega \wedge \omega = \omega \cdot \omega \). This means that \( \omega^m \) is a scalar if \( m \) is even and \( \omega^m \) is a vector if \( m \) is odd.

### 6. Application of the QFT

In this section, we apply the QFT to partial differential equations in quaternion algebra (compare to Obolashvili\(^{20}\)). We consider the following initial value problem:
\[ \frac{\partial u}{\partial t} - \nabla^2 u = 0, \quad \text{on } \mathbb{R}^{0,2} \times (0, \infty) \] (6.1)
and
\[ u(\mathbf{x}, 0) = f(\mathbf{x}), \quad f \in \mathcal{S}(\mathbb{R}^{0,2}; \mathbb{H}), \] (6.2)
where \( \mathcal{S}(\mathbb{R}^{0,2}; \mathbb{H}) \) is the quaternion Schwartz space, that is, the set of rapidly decreasing functions from \( \mathbb{R}^{0,2} \) to \( \mathbb{H} \). Applying the definition of the QFT to both sides of (6.1) with respect to \( \mathbf{x} \) and using (5.11), we have
\[ \mathcal{F}_q \{ u \}(\omega) = \mathcal{F}_q \{ f \}(\omega) \left( \frac{i + j + k}{\sqrt{3}} \omega_1 \right)^2 + \mathcal{F}_q \{ u \}(\omega) \left( \frac{i + j + k}{\sqrt{3}} \omega_2 \right)^2 \]
\[ = - (\omega_1^2 + \omega_2^2) \mathcal{F}_q \{ u \}(\omega). \] (6.3)

Assume that \( u(\mathbf{x}, t) \) is sufficiently nice to allow the interchange of differentiation with respect to \( t \) and the QFT, that is, \( \mathcal{F}_q \{ \frac{\partial u}{\partial t} \} = \frac{\partial}{\partial t} \mathcal{F}_q \{ u \} \). Then, the general solution of (6.3) is given by
\[ \mathcal{F}_q \{ u \}(\omega, t) = Ce^{-(\omega_1^2 + \omega_2^2)t}, \] (6.4)
where \( C \) is a quaternion constant. We impose the initial condition \( \mathcal{F}_q \{ u \}(\omega, 0) = \mathcal{F}_q \{ f \}(\omega) \) to obtain
\[ \mathcal{F}_q \{ u \}(\omega, t) = e^{-(\omega_1^2 + \omega_2^2)t} \mathcal{F}_q \{ f \}(\omega). \] (6.5)

Note that the QFT of a Gaussian quaternion function is also a Gaussian quaternion function (compare to Bahri, Hitzer, Hayashi and Ashino\(^7\)). More precisely, we have
\[ \frac{1}{4\pi t} \mathcal{F}_q \{ e^{-((\omega_1^2 + \omega_2^2)/4t)} \} = e^{-(\omega_1^2 + \omega_2^2)t}. \] (6.6)

Applying the inverse QFT, we have
\[ u(\mathbf{x}, t) = \mathcal{F}_q^{-1} \left[ e^{-(\omega_1^2 + \omega_2^2)t} \mathcal{F}_q \{ f \} \right](\mathbf{x}) \]
\[ = \mathcal{F}_q^{-1} \left[ \frac{1}{4\pi t} \mathcal{F}_q \{ e^{-(\omega_1^2 + \omega_2^2)/4t} \} \mathcal{F}_q \{ f \} \right](\mathbf{x}). \] (6.7)
Since
\[ \mathcal{F}_q \{ e^{-((x_1^2 + x_2^2)/4t)} \} \{ \omega \} = 4\pi t e^{-(\omega_1^2 + \omega_2^2)t} \in \mathbb{R}, \]
then we can apply the convolution theorem (4.8) to have
\[ u(x, t) = K_t(x) \ast f, \quad (6.8) \]
where \( K_t(x) = \frac{1}{4\pi t} e^{-((x_1^2 + x_2^2)/4t)} \). If we decompose \( f = f_0 + if_1 + jf_2 + kf_3 \), then Eq. (6.8) reduces to
\[ u(x, t) = K_t(x) \ast f_0 + iK_t(x) \ast f_1 + jK_t(x) \ast f_2 + kK_t(x) \ast f_3, \quad (6.9) \]
where \( f_i \in \mathbb{R}, i = 0, 1, 2, 3 \). By Definition 4.1 of convolution, (4.4) gives
\[
\begin{align*}
    u(x, t) &= \frac{1}{4\pi t} \int_{\mathbb{R}^2} e^{-((x-y)^2/(4t))} f_0(y) \, d^2y + \frac{i}{4\pi t} \int_{\mathbb{R}^2} e^{-((x-y)^2/(4t))} f_1(y) \, d^2y \\
    &\quad + \frac{j}{4\pi t} \int_{\mathbb{R}^2} e^{-((x-y)^2/(4t))} f_2(y) \, d^2y + \frac{k}{4\pi t} \int_{\mathbb{R}^2} e^{-((x-y)^2/(4t))} f_3(y) \, d^2y.
\end{align*}
\]
(6.10)
Thus, we have the following theorem.

**Theorem 6.1.** Let \( f_i, i = 0, 1, 2, 3, \) belong to \( L^p(\mathbb{R}^2; \mathbb{R}) \), \( 1 \leq p \leq \infty \). Put \( u_i(x, t) = K_t(x) \ast f_i \). Then, each \( u_i(x, t), i = 0, 1, 2, 3, \) is a solution of
\[ \frac{\partial u_i}{\partial t} - \nabla^2 u_i = 0 \quad \text{on} \quad \mathbb{R}^2 \times (0, \infty) \]
and
\[ u(x, t) = u_0(x, t) + iu_1(x, t) + ju_2(x, t) + ku_3(x, t) \]
is a solution of (6.1).

**Proof.** It is well-known that each \( u_i(x, t) = K_t(x) \ast f_i \) satisfies the heat equation. Using the superposition principle, we have that
\[ K_t(x) \ast f_0 + iK_t(x) \ast f_1 + jK_t(x) \ast f_2 + kK_t(x) \ast f_3 \quad (6.11) \]
is a solution of the quaternion heat equation (6.1).

A similar argument gives the following theorem.

**Theorem 6.2.** Let \( f_i, i = 0, 1, 2, 3, \) be bounded and continuous. Then, we have the following:

(i) Each \( u_i(x, t) = K_t(x) \ast f_i, i = 0, 1, 2, 3, \) is continuous on \( \mathbb{R}^2 \times (0, \infty) \) and satisfies \( u_i(x, 0) = f_i(x) \).

(ii) The solution \( u \) is continuous on \( \mathbb{R}^2 \times (0, \infty) \) and satisfies \( u(x, 0) = f(x) \).
7. **Construction of 2D QWT**

To extend the classical wavelet theory to quaternion algebra, we use the kernel of the QFT in Definition 3.1 (compare to Bahri, Ashino and Vaillancourt\(^4\)). This section constructs the 2D CQWT.

### 7.1. **Admissible quaternion wavelet**

Denote by \(SO(n)\), \(n \in \mathbb{N}\), the special orthogonal group, that is, the rotation group in \(\mathbb{R}^n\).

**Definition 7.1 (Admissible quaternion wavelet).** A pair \(\{\psi, \phi\}\) of functions in \(L^2(\mathbb{R}^2; \mathbb{H})\) is said to be an admissible quaternion wavelet (AQW) pair if \(\psi\) and \(\phi\) satisfy the following admissibility condition:

\[
C_{\{\psi, \phi\}} = \int_{SO(2)} \int_{\mathbb{R}^+} \frac{\hat{\psi}(ar_\theta(\omega))\hat{\phi}(ar_\theta(\omega))}{a} da d\theta
\]  

is a nonzero quaternion constant independent of \(\omega\) satisfying \(|\omega| = 1\). Denote by AQW, the class of admissible quaternion wavelet pairs \(\{\psi, \phi\}, \psi, \phi \in L^2(\mathbb{R}^2; \mathbb{H})\).

A function \(\psi \in L^2(\mathbb{R}^2; \mathbb{H})\) is said to be an admissible quaternion wavelet if \(\psi\) satisfies the following admissibility condition:

\[
C_{\psi} = \int_{SO(2)} \int_{\mathbb{R}^+} |\hat{\psi}(ar_\theta(\omega))|^2 \frac{da d\theta}{a}
\]  

is a real positive constant independent of \(\omega\) satisfying \(|\omega| = 1\).

As a matter of convenience, we denote \(\psi \in AQW\) when \(\psi\) is an admissible quaternion wavelet.

A \(\psi \in AQW\) is sometimes called a quaternion mother wavelet and \(\psi_{a, \theta, b}(x)\) are called daughter quaternion wavelets, where \(a\) is a dilation parameter, \(b\) is a translation vector parameter and \(\theta\) is an \(SO(2)\) rotation parameter.

**Remark 7.1.** If \(|\hat{\psi}(\xi)|\) is continuous near \(\xi = 0\), then the existence of integral (7.2) guarantees that \(\hat{\psi}(0) = 0\). Since the QFT \(\hat{\psi}\) of each quaternion mother wavelet \(\psi \in L^1 \cap L^2(\mathbb{R}^2; \mathbb{H})\) is bounded and continuous, we have

\[
\hat{\psi}(0) = \int_{\mathbb{R}^2} (\psi_0(x) + i\psi_1(x) + j\psi_2(x) + k\psi_3(x))e^{-\frac{i\xi_0 x}{\sqrt{3}}} d^2 x
\]

\[
= \int_{\mathbb{R}^2} \psi_0(x) d^2 x + i \int_{\mathbb{R}^2} \psi_1(x) d^2 x
\]

\[
+ j \int_{\mathbb{R}^2} \psi_2(x) d^2 x + k \int_{\mathbb{R}^2} \psi_3(x) d^2 x = 0.
\]

This means that the integral of every component \(\psi_s\) of the quaternion mother wavelet is zero:

\[
\int_{\mathbb{R}^2} \psi_s(x) d^2 x = 0, \quad s = 0, 1, 2, 3,
\]  

where \(\psi_s \in L^1 \cap L^2(\mathbb{R}^2; \mathbb{R}), s = 0, 1, 2, 3\).
Defining quaternion mother wavelets

**Definition 7.2.** For $ψ ∈ L^2(\mathbb{R}^2; \mathbb{H})$, $a ∈ \mathbb{R}^+ = \{a ∈ \mathbb{R} | a > 0\}$, $b ∈ \mathbb{R}^2$ and $r_0 ∈ SO(2)$, we define the unitary linear operator

$$U_{a,θ,b} : L^2(\mathbb{R}^2; \mathbb{H}) → L^2(\mathcal{G}; \mathbb{H}),$$

by

$$(U_{a,θ,b}(ψ)) = ψ_{a,θ,b}(x) = \frac{1}{a}ψ\left(r_θ \left(\frac{x - b}{a}\right)\right).$$  \hspace{1cm} (7.4)

Here, we denote by $\mathcal{G}$, the similarity group SIM(2), which is defined by

$$SIM(2) = \mathbb{R}^+ × SO(2) × \mathbb{R}^2$$

$$= \{(a, r_θ, b) | a ∈ \mathbb{R}^+, r_θ ∈ SO(2), b ∈ \mathbb{R}^2\}. \hspace{1cm} (7.5)$$

For an admissible pair $\{ψ, ψ\} ∈ A \omega Q$, $ψ$ is called a quaternion mother wavelet and $ψ_{a,θ,b}(x)$ are called daughter quaternion wavelets, where $a$ is a dilation parameter, $b$ is a translation vector parameter and $θ$ is an $SO(2)$ rotation parameter.

Straightforward calculations give the following lemma.

**Lemma 7.1.** Let $ψ$ be a quaternion mother wavelet. The daughter quaternion wavelets (7.4) can be represented in terms of the QFT as

$$\mathcal{F}_q\{ψ_{a,θ,b}\}(ω) = a\tilde{ψ}(ar_θω)e^{-\frac{ix + jy + kω}{\sqrt{a}}b}. \hspace{1cm} (7.6)$$

**Proof.** By definition, we have

$$\mathcal{F}_q\{ψ_{a,θ,b}\}(ω) = \int_{\mathbb{R}^2} \frac{1}{a}ψ\left(r_θ \left(\frac{x - b}{a}\right)\right)e^{-\frac{ix + jy + kω}{\sqrt{a}}x} d^2x. \hspace{1cm} (7.7)$$

Performing the change of variables $\frac{x - b}{a} = y$ into the above expression, we have

$$\mathcal{F}_q\{ψ_{a,θ,b}\}(ω) = a \int_{\mathbb{R}^2} ψ(r_θy)e^{-\frac{ix + jy + kω}{\sqrt{a}}(b + ay)} \frac{1}{a^2} d^2y$$

$$= a \int_{\mathbb{R}^2} ψ(r_θy)e^{-\frac{ix + jy + kω}{\sqrt{a}}b} e^{-\frac{ix + jy + kω}{\sqrt{a}}y} d^2y$$

$$= a \int_{\mathbb{R}^2} ψ(r_θy)e^{-\frac{ix + jkω}{\sqrt{a}}ω} d^2y e^{-\frac{ix + jkω}{\sqrt{a}}b}.$$ 

Putting $z = r_θy$, we easily obtain

$$\mathcal{F}_q\{ψ_{a,θ,b}\}(ω) = a \int_{\mathbb{R}^2} ψ(z)e^{-\frac{ix + jkω}{\sqrt{a}}ω(r_θz)} \det(r_θ) d^2z e^{-\frac{ix + jkω}{\sqrt{a}}b}$$

$$= a \int_{\mathbb{R}^2} ψ(z)e^{-\frac{ix + jkω}{\sqrt{a}}ω(r_θz)} d^2ze^{-\frac{ix + jkω}{\sqrt{a}}b}$$

$$= a\tilde{ψ}(ar_θω)e^{-\frac{ix + jkω}{\sqrt{a}}b}, \hspace{1cm} (7.8)$$

which was to be proved.

\[\Box\]
7.2. 2D continuous quaternion wavelet transform

Definition 7.3 (CQWT). The 2D CQWT,

\[ T_\psi : L^2(\mathbb{R}^2; \mathbb{H}) \to L^2(\mathbb{R}^2; \mathbb{H}) \]

of a quaternion-valued function \( f \in L^2(\mathbb{R}^2; \mathbb{H}) \) with respect to a quaternion mother wavelet \( \psi \) is defined by

\[ f \mapsto T_\psi f(a, \theta, b) = (f, \psi_{a, \theta, b})_{L^2(\mathbb{R}^2; \mathbb{H})} \]

\[ = \int_{\mathbb{R}^2} f(x) \overline{\frac{1}{a} \psi \left( r - \frac{\theta}{a} \left( \frac{x - b}{a} \right) \right)} \, d^2x. \quad (7.9) \]

It must be remarked that the order of the terms in (7.9) is fixed because of the noncommutativity of the product of quaternions. Changing the order yields another quaternion-valued function which differs by the signs of the terms. Equation (7.9) clearly shows that the CQWT can be regarded as the inner product of a quaternion-valued signal \( f \) with a daughter quaternion-wavelet.

Lemma 7.2. Let \( \psi \) be a quaternion mother wavelet. Then, the CQWT of \( \psi \) has a quaternion Fourier representation of the form

\[ T_\psi f(a, \theta, b) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} a \hat{f}(\omega) e^{\frac{i+j+k}{\sqrt{3}} \omega \cdot b} \overline{\psi(\omega)} \, d^2\omega. \quad (7.10) \]

Proof. Simple computations yield

\[ T_\psi f(a, \theta, b) = (f, \psi_{a, \theta, b})_{L^2(\mathbb{R}^2; \mathbb{H})} \]

\[ = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\omega) \overline{\psi_{a, \theta, b}(\omega)} \, d^2\omega \]

\[ \overset{(4.1)}{=} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\omega) \overline{\psi_{a, \theta, b}(\omega)} \, d^2\omega \]

\[ \overset{(7.6)}{=} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} a \hat{f}(\omega) \overline{\psi(\omega)} e^{\frac{i+j+k}{\sqrt{3}} \omega \cdot b} \, d^2\omega \]

\[ \overset{(2.5)}{=} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} a \hat{f}(\omega) e^{\frac{i+j+k}{\sqrt{3}} \omega \cdot b} \overline{\psi(\omega)} \, d^2\omega. \quad (7.11) \]

This proves (7.10).

We now deduce the following proposition from Lemma 7.2.

Proposition 7.1. Let \( \psi \) be a quaternion mother wavelet. If we assume that

\[ e^{\frac{i+j+k}{\sqrt{3}} \omega \cdot b} \overline{\psi(\omega)} = \psi(\omega) e^{\frac{i+j+k}{\sqrt{3}} \omega \cdot b}, \]

then

\[ T_\psi f(a, \theta, b) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} a \hat{f}(\omega) \psi(\omega) \, d^2\omega. \]
then Eq. (7.10) has the form

\[
T_\psi f(a, \theta, b) = \frac{a}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\omega) \overline{\psi(a \omega - \theta)} e^{i \frac{\omega + b^T \omega}{3} \omega} d^2 \omega
\]

\[
= \mathcal{F}_q^{-1}\{a \hat{f}(\omega) \overline{\psi(a \omega - \theta)}\}(b). \tag{7.12}
\]

Or, equivalently,

\[
\mathcal{F}_q(T_\psi f(a, \theta, \cdot))(\omega) = a \hat{f}(\omega) \overline{\psi(a \omega - \theta)}. \tag{7.13}
\]

**Remark 7.2.** Let \( \omega_0 = u e_1 + v e_2 \) be an arbitrary frequency vector, and \( \xi(x) = e^{-\frac{1}{4} \omega_0^T x} e^{-\frac{1}{2} x^2} \) a correction term in order for Eq. (7.3) to be satisfied. Then, the quaternionic Gabor wavelet is given by (compare to Bahri and Hitzer)

\[
\psi^0(x) = (e^{i \frac{\omega_0 + b^T \omega_0}{3} \omega} e^{-\frac{1}{2} x^2}) - \xi(x). \tag{7.14}
\]

Using the modulation property of the QFT, we can represent the quaternionic Gabor wavelet (7.14) in terms of the QFT as

\[
\mathcal{F}_q(\psi^0)(\omega) = e^{-\frac{1}{4} (\omega - \omega_0)^2} - e^{-\frac{1}{2} (\omega^2 - \omega_0^2)}. \tag{7.15}
\]

It is easy to show that the quaternionic Gabor wavelet satisfies the assumption of Preposition 7.1.

Remark 7.2 leads to the following remark.

**Remark 7.3.** It is not difficult to check that Proposition 7.1 holds if the quaternion mother wavelet is of the form

\[
\mathcal{F}_q(\psi)(\omega) = \mathcal{F}_q(\psi_0)(\omega) + (i + j + k) \mathcal{F}_q(\psi_1)(\omega), \quad \mathcal{F}_q(\psi_0), \mathcal{F}_q(\psi_1) \in \mathbb{R}. \tag{7.16}
\]

### 7.3. Reproducing formula

In this section, we show that a quaternion function, \( f \), can be recovered from its CQWT whenever the quaternion wavelets satisfy the following admissibility condition.

**Theorem 7.1 (Inner product relation).** Let \( \{\psi, \phi\} \in AQW \), where

\[
\psi = \psi_0 + i \psi_1 + j \psi_2 + k \psi_3, \quad \phi = \phi_0 + i \phi_1 + j \phi_2 + k \phi_3.
\]

Assume that both quaternion mother wavelets \( \psi \) and \( \phi \) satisfy the assumption of Preposition 7.1. Then, for every \( f, g \in L^1 \cap L^2(\mathbb{R}^2; \mathbb{H}) \), we have

\[
\int_{SO(2)} \int_{\mathbb{R}^+} \left( \int_{\mathbb{R}^2} T_\psi f(a, \theta, b) \overline{T_\phi g(a, \theta, b)} d^2 b \right) \frac{da d\theta}{a^3} = (f C_{\{\psi, \phi\}}, g)_{L^2(\mathbb{R}^2; \mathbb{H})}. \tag{7.17}
\]
Corollary 7.1. Let $(\psi, \psi) \in AQW$. Then we have

\[(T_{\psi} f, T_{\psi} g)_{L^2(\mathbb{R}^2; \mathbb{H})} = C_{\psi}(f, g)_{L^2(\mathbb{R}^2; \mathbb{H})}.\]
Remark 7.4. Note that (7.20) is of the same form as the inner product relation of the classical WT (see Debnath\cite{Debnath10} and Mallat\cite{Mallat19}). In particular, if $f = g$ in Corollary 7.1 we have the \textit{CQWT Plancherel Formula}:

\[
\| T_\phi f \|^2_{L^2(\mathbb{H}; \mathbb{H})} = C_\phi \| f \|^2_{L^2(\mathbb{H}; \mathbb{H})}. \tag{7.21}
\]

Theorem 7.2 (Inversion formula). Let $\{ \psi, \phi \} \in AQP$. Then, $f \in L^2(\mathbb{R}^2; \mathbb{H})$ can be reconstructed from $T_\phi f(a, b, \theta)$ by

\[
f(x) = \frac{1}{|C_{\psi, \phi}|^2} \int G T_\psi f(a, b, \theta) \phi_{a, b, \theta} C_{\psi, \phi} \frac{da \, d\theta}{a^3} \, d^2b, \tag{7.22}
\]

where the integral converges in the weak sense.

Proof. Applying Theorem 7.1 to $g \in L^2(\mathbb{R}^2; \mathbb{H})$, we have

\[
(fC_{\psi, \phi}, g)_{L^2(\mathbb{R}^2; \mathbb{H})} = \int_{SO(2)} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} T_\psi f(a, \theta, b) T_\phi g(a, \theta, b) \, d^2b \right) \frac{da \, d\theta}{a^3} \, d^2b,
\]

\[
= \int_G \int_{\mathbb{R}^2} T_\psi f(a, \theta, b) \phi_{a, \theta, b}(x) \overline{g(x)} \, d^2x \frac{da \, d\theta}{a^3} \, d^2b
\]

\[
= \int_G \int_{\mathbb{R}^2} T_\psi f(a, \theta, b) \phi_{a, \theta, b}(x) \overline{g(x)} \, d^2x \frac{da \, d\theta}{a^3} \, d^2b \, d^2x
\]

\[
= \left( \int_G T_\psi f(a, \theta, b) \phi_{a, \theta, b} \frac{da \, d\theta}{a^3} \, d^2b, g \right)_{L^2(\mathbb{R}^2; \mathbb{H})}. \tag{7.23}
\]

Since the inner product identity holds for every $g \in L^2(\mathbb{R}^2; \mathbb{H})$, we conclude that

\[
f(x)C_{\psi, \phi} = \int_G T_\psi f(a, b, \theta) \phi_{a, b, \theta}(x) \frac{da \, d\theta}{a^3} \, d^2b,
\]

\[
f(x) = \int_G T_\psi f(a, b, \theta) \phi_{a, b, \theta}(x) C_{\psi, \phi}^{-1} \frac{da \, d\theta}{a^3} \, d^2b, \tag{7.24}
\]

which implies (7.22) by (2.8).

8. Conclusion

Using basic quaternion properties, we introduced the QFT. Because of the noncommutativity of the multiplication in the quaternion space $\mathbb{H}$, several important properties of the classical FT, such as modulation and convolution, must be modified. We also studied a simple application of the QFT to partial differential equations in quaternion algebra. Using the kernel of the QFT, we proposed a 2D continuous quaternion wavelet transform (CQWT). We extended to the CQWT some fundamental properties of the continuous WT, such as the inner product and the...
inversion formula. This generalization also enables us to establish the quaternionic Gabor wavelets, which can go beyond the application area of the two-dimensional complex Gabor wavelets.

Acknowledgment

Thanks are due to the anonymous referees for their deep remarks which considerably improved this paper. This work was partially supported by Bantuan Seminar Luar Negeri oleh DP2M DIKTI (S-4334/PB/2011) Indonesia, JSPS.KAKENHI (C)22540130 of Japan and NSERC of Canada.

References

10. L. Debnath, Wavelet Transforms and Their Applications (Birkhäuser, Boston, 2002).
Continuous quaternion Fourier and wavelet transforms