Nonlinear transient heat conduction problems for a class of inhomogeneous anisotropic materials by BEM

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Received 20 March 2007; accepted 26 April 2007
Available online 20 February 2008

Abstract

A class of nonlinear transient heat conduction problems for anisotropic inhomogeneous materials is considered. The problems under consideration are reduced to a boundary integral equation which may be solved numerically using standard techniques. Some numerical examples are considered in order to test the accuracy of the numerical procedure.

Keywords: Boundary integral methods; Heat transfer; Nonlinear differential equations; Numerical methods

1. Introduction

Boundary integral equation methods have been used by a number of authors to consider steady-state and transient heat conduction problems for inhomogeneous materials. For example Divo and Kassab in [1] considered the linear elliptic equation governing the steady-state temperature field in inhomogeneous isotropic media while Clements et al. [2] obtained a boundary integral equation procedure which may be used to determine the steady-state temperature field for a wide class of inhomogeneous anisotropic materials. For time dependent problems Abdullah et al. [3] and Sutradhar et al. [4] employed the Laplace transform and a boundary integral formulation to solve problems for particular classes of isotropic inhomogeneous materials.

Nonlinear heat conduction problems have been investigated by, for example, Clements and Budhi [5] who considered steady-state problems for a class of materials for which the thermal conductivity varied with the temperature and the three spatial coordinates. In the case of transient nonlinear problems boundary integral equation methods have been employed by a number of authors including Bialecki and Kuhn [6], Goto and Suzuki [7] and Kikuta et al. [8].

This paper is concerned with nonlinear transient heat conduction problems for a class of anisotropic inhomogeneous materials. An outline of the paper is as follows. The Kirchhoff transformation is employed to convert the governing nonlinear equation to a linear second order partial differential equation with variable coefficients. A further transformation is then applied in order to transform the variable coefficients equation to a linear equation with constant coefficients. A boundary integral equation formulation is obtained for the solution of the constant coefficients equation. Then use of the inverse transformation provides a boundary integral equation for the solution of the linear variable coefficients equation. Once numerical values have been obtained from this boundary integral equation use of the inverse of the Kirchhoff transformation provides the temperature field for particular boundary value problems.

Essentially the present work may be considered to be a generalisation of the paper of Clements and Budhi [5] to include transient effects and, more particularly, a generalisation of the paper by Goto and Suzuki [7] to include a restricted class of thermal coefficients which vary with position.
2. Basic equations

Referred to the Cartesian frame \( O_{x_1,x_2,x_3} \) this paper is concerned with obtaining the solution of the heat conduction problems governed by the equation

\[
\frac{\partial}{\partial x_i} \left[ k_{ij}(x, T) \frac{\partial T(x, t)}{\partial x_j} \right] = \rho(x, T)c(x, T) \frac{\partial T(x, t)}{\partial t}
\]

for \( i, j = 1, 2, 3 \),

\( 1 \)

where \( T \) is the temperature, \( k_{ij} \) are the heat conductivities, \( \rho \) is the density, \( c \) denotes the heat capacity, \( x = (x_1, x_2, x_3) \), \( t \) is the time variable and the repeated suffix summation convention is employed. The analysis employed by Goto and Suzuki in \([7]\) is used and extended to derive a boundary element method (BEM) for the solution of (1).

The matrix of the heat conductivities \( k_{ij} \) is assumed to be a real symmetric positive definite matrix. Also the heat conductivities \( k_{ij} \) are assumed to take the form

\[
k_{ij}(x, T) = \mu_{ij}(x)h(T),
\]

\( 2 \)

where \( \mu_{ij} \) are differentiable functions and \( h \) is an integrable function.

The analysis is specially relevant to an anisotropic medium but it equally applies to the isotropic case. For isotropic media the coefficients \( k_{ij} \) in (1) take the form \( k_{11} = k_{22} = k_{33} \) and \( k_{12} = k_{13} = k_{23} = 0 \) and use of these equations in the following analysis immediately yields the corresponding results for isotropic materials.

3. The initial-boundary value problem

A solution to (1) is sought in a domain \( \Omega \) for time \( t \geq 0 \). The boundary \( \Gamma \) of \( \Omega \) consists of a finite number of piecewise smooth closed surfaces. In \( \Omega \) the initial temperature \( T(x, 0) \) is given (see Fig. 1). On \( \Gamma' \) the temperature \( T(x, t) \) is specified and on \( \Gamma'' \) the flux \( P_T(x, t) \) is specified

where \( \Gamma = \Gamma' \cup \Gamma'' \), \( P_T = k_{ij}(\partial T/\partial x_j)n_i \) with \( n_i \) denoting the \( i \)th component of the outward pointing vector \( n \) normal to \( \Gamma \).

Substitution of (2) into (1) yields

\[
\frac{\partial}{\partial x_i} \left[ \mu_{ij}(x)h(T) \frac{\partial T(x, t)}{\partial x_j} \right] = \rho(x, T)c(x, T) \frac{\partial T(x, t)}{\partial t}.
\]

\( 3 \)

Use of the Kirchhoff’s transformation

\[
\Theta = \int h(T) \, dT
\]

\( 4 \)

in (3) gives

\[
\frac{\partial}{\partial x_i} \left[ \mu_{ij}(x) \frac{\partial \Theta(x, t)}{\partial x_j} \right] = \left[ \rho(x, \Theta)c(x, \Theta)/h(\Theta) \right] \frac{\partial \Theta(x, t)}{\partial t}.
\]

\( 5 \)

The corresponding initial and boundary conditions for (5) are

\[
\Theta(x, 0) = \Theta(T(x, 0)) \quad \text{for } x \in \Omega,
\]

\[
\Theta(x, t) = \Theta(T(x, t)) \quad \text{for } x \in \Gamma',
\]

\( 6 \)

\[
P_{\Theta}(x, t) = P_T(x, t) \quad \text{for } x \in \Gamma'',
\]

\( 7 \)

where

\[
P_{\Theta} = \mu_{ij}(\partial \Theta/\partial x_j)n_i.
\]

4. Integral equation formulation

The coefficients \( \mu_{ij}(x) \) are required to take the form

\[
\mu_{ij}(x) = \kappa_{ij}g(x),
\]

\( 8 \)

where the \( \kappa_{ij} \) are constants and \( g(x) \) is required to satisfy

\[
\kappa_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = 0.
\]

\( 9 \)

Substitution of (8) and use of the transformation

\[
\psi(x, t) = g^{1/2}(x)\Theta(x, t)
\]

\( 10 \)

in (5) yields

\[
\kappa_{ij} \frac{\partial^2 \psi(x, t)}{\partial x_i \partial x_j} - C(x, \psi) \frac{\partial \psi(x, t)}{\partial t} = 0,
\]

\( 11 \)

where

\[
C(x, \psi) = \frac{\rho(x, \psi)c(x, \psi)}{h(\psi)g(x)}.
\]

\( 12 \)

Let \( G(x, t, \xi, \tau) \) be the fundamental solution of the equation

\[
C_r \frac{\partial G(x, t, \xi, \tau)}{\partial t} - \kappa_r \frac{\partial^2 G(x, t, \xi, \tau)}{\partial x_i \partial x_j} = 0,
\]

\( 13 \)

where \( C_r \) is a constant reference value of \( C(x, \psi) \). Hence

\[
C_r \frac{\partial G(x, t, \xi, \tau)}{\partial t} - \kappa_r \frac{\partial^2 G(x, t, \xi, \tau)}{\partial x_i \partial x_j} = \delta(x - \xi)\delta(t - \tau),
\]

\( 14 \)

\[
C_r \frac{\partial G(x, t, \xi, \tau)}{\partial t} + \kappa_r \frac{\partial^2 G(x, t, \xi, \tau)}{\partial \xi^2} = -\delta(x - \xi)\delta(t - \tau).
\]

\( 15 \)
Multiply (11) by $G(x, t|\xi, \tau)$ and integrate
\[
\int_t^T \int_\Omega [G(x, t|\xi, \tau)] \left( C(x, \psi) \frac{\partial \psi(x, \xi, \tau)}{\partial \tau} - \kappa_{ij} \frac{\partial^2 \psi(x, \xi, \tau)}{\partial \xi_i \partial \xi_j} \right) \, d\Omega(\xi) \, d\tau = 0
\]
and multiply (15) by $\psi(x, \xi, \tau)$ and integrate
\[
-\psi(x, t) = \int_t^T \int_\Omega \psi(x, \xi, \tau) \left[ C \frac{\partial G(x, t|\xi, \tau)}{\partial \tau} \right. \\
+ \kappa_{ij} \frac{\partial^2 G(x, t|\xi, \tau)}{\partial \xi_i \partial \xi_j} \] \\
\left. + C \psi(x, t|\xi, \tau) \right] \, d\Omega(\xi) \, d\tau.
\]
Adding (16) to (17) yields
\[
-\psi(x, t) = \int_t^T \int_\Omega \left[ C(x, \psi) G(x, t|\xi, \tau) \frac{\partial \psi(x, \xi, \tau)}{\partial \tau} \\
+ C \psi(x, t|\xi, \tau) \right. \\
+ C \psi(x, t|\xi, \tau) \bigg] \, d\Omega(\xi) \, d\tau.
\]
Now
\[
\frac{\partial}{\partial \tau} [G(x, t|\xi, \tau) \psi(x, \xi, \tau)] = \psi(x, \xi, \tau) \frac{\partial G(x, t|\xi, \tau)}{\partial \tau} \\
+ G(x, t|\xi, \tau) \frac{\partial \psi(x, \xi, \tau)}{\partial \tau}
\]
so that
\[
\psi(x, \xi, \tau) \frac{\partial G(x, t|\xi, \tau)}{\partial \tau} = -G(x, t|\xi, \tau) \frac{\partial \psi(x, \xi, \tau)}{\partial \tau} \\
+ \frac{\partial}{\partial \tau} [G(x, t|\xi, \tau) \psi(x, \xi, \tau)].
\]
Use of (19) in (18) provides
\[
-\psi(x, t) = \int_t^T \int_\Omega \left[ C(x, \psi) - C_r \right] G(x, t|\xi, \tau) \frac{\partial \psi(x, \xi, \tau)}{\partial \tau} \, d\Omega(\xi) \, d\tau \\
+ \int_t^T \int_\Omega \left[ \psi(x, \xi, \tau) \kappa_{ij} \frac{\partial^2 G(x, t|\xi, \tau)}{\partial \xi_i \partial \xi_j} \\
- G(x, t|\xi, \tau) \kappa_{ij} \frac{\partial^2 \psi(x, \xi, \tau)}{\partial \xi_i \partial \xi_j} \right] \, d\Omega(\xi) \, d\tau \\
+ \int_t^T \int_\Omega C_r \frac{\partial}{\partial \tau} [G(x, t|\xi, \tau) \psi(x, \xi, \tau)] \, d\Omega(\xi) \, d\tau.
\]
Now
\[
\int_0^T \int_\Omega C_r \frac{\partial}{\partial \tau} [G \psi] \, d\Omega(\xi) \, d\tau = C_r \int_\Omega d\Omega(\xi) \int_0^T \frac{\partial}{\partial \tau} [G \psi] \, d\tau \\
= -C_r \int_\Omega [G \psi]_{\xi=0} \, d\Omega(\xi)
\]
since $G(x, t|\xi, \tau) = 0$ when $\tau = t$. Also Green's theorem gives
\[
\int_\Omega \left[ \psi \kappa_{ij} \frac{\partial^2 G}{\partial \xi_i \partial \xi_j} - G \psi \kappa_{ij} \frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j} \right] \, d\Omega(\xi) \\
= \int_\Gamma [\psi P_G - GP_{\psi}] \, d\Gamma(\xi),
\]
where
\[
P_G(x, t|\xi, \tau) = \kappa_{ij} \frac{\partial^2 G(x, t|\xi, \tau)}{\partial \xi_i \partial \xi_j} n_j,
\]
and
\[
P_{\psi}(x, t|\xi, \tau) = \kappa_{ij} \frac{\partial \psi(x, \xi, \tau)}{\partial \xi_i} n_j.
\]
Use of (21) and (22) in (20) gives
\[
\eta(x) \psi(x, t) = \int_t^T \int_\Omega [G(x, t|\xi, \tau) \psi(x, \xi, \tau) \\
- \psi(x, \xi, \tau) P_{\partial G(x, t|\xi, \tau)]} \, d\Gamma(\xi) \, d\tau \\
- \int_t^T \int_\Omega \left[ C(x, \psi) - C_r \right] G(x, t|\xi, \tau) \\
\times \frac{\partial \psi(x, \xi, \tau)}{\partial \tau} \, d\Omega(\xi) \, d\tau \\
+ \int_\Omega C_r \left[ G(x, t|\xi, \tau) \psi(x, \xi, \tau) \right] \, d\Omega(\xi),
\]
where $\eta = 0$ if $x \notin \Omega \cup \Gamma$, $\eta = 1$ if $x \in \Omega$, $\eta = \frac{1}{2}$ if $x \in \Gamma$ and $\Gamma$ has a continuously turning tangent at $x$.

Consider the function
\[
H(x, \psi) = \int_\Omega [C(x, \psi) - C_r] \, d\psi,
\]
\[
\frac{\partial H}{\partial \psi} \frac{\partial \psi}{\partial \tau} = (C_r - \frac{\partial \psi}{\partial \tau}).
\]
Substitution of (27) into the second integral in (25) yields
\[
\int_t^T \int_\Omega (C_r - \frac{\partial \psi}{\partial \tau}) \frac{\partial \psi}{\partial \tau} \, d\Omega(\xi) \, d\tau \\
= \int_t^T \int_\Omega \frac{\partial H}{\partial \psi} G \, d\Omega(\xi) \, d\tau.
\]
Use of integration by parts for the integral on the right-hand side of (28) gives
\[
\int_0^T \frac{\partial H}{\partial \tau} \, G \, d\tau = \int_0^T \frac{\partial G}{\partial \tau} \, d\tau.
\]
The unknowns are now assumed to be constant, on each time element. Hence, following the work of Goto and Suzuki in [7], if the unknown $\psi$ (and hence the temperature $T$) is assumed to take on its value at the upper bound ($\tau = t$) of the semi-open time interval $(0, t]$ then
\[
\int_0^T \frac{\partial G}{\partial \tau} \, d\tau \approx [H]_{\tau=t} \frac{G_{\tau=t}}{G_{\xi=0}}.
\]
Substitution of (30) into (29) provides
\[
\int_0^T \frac{\partial H}{\partial \tau} \, G \, d\tau \approx [H]_{\tau=t} \frac{G_{\tau=t}}{G_{\xi=0}} - [H]_{\tau=0} \frac{G_{\tau=0}}{G_{\xi=0}}
\]
\[
= [H]_{\tau=t} \frac{G_{\tau=t}}{G_{\xi=0}}.
\]
So the integral equation (25) may be written in the form
\[
\eta(x)\psi(x, t) = \int_{\Omega} \left[ C_{\tau}\psi(\xi, 0) - H(\xi, \psi(\xi, t)) \right]
+ \int_{\Gamma} \left[ P_{\phi}(\xi, t) \int_{0}^{t} G(x, t, \xi, \tau) d\tau \right]
- \psi(\xi, 0) \int_{0}^{t} P_{G}(x, t, \xi, \tau) d\tau \right] \right] d\Gamma(\xi).
\tag{32}
\]

However, if the temperature is assumed to take on its values at the lower bound \(\tau = 0\) of the semi-open time interval \([0, t]\) then
\[
\int_{0}^{t} \frac{\partial H}{\partial \tau} G \, d\tau = 0.
\tag{33}
\]

So the integral equation (25) may be written in the form
\[
\eta(x) \psi(x, t) = \int_{\Omega} \left[ P_{\phi}(\xi, 0) \int_{0}^{t} G(x, t, \xi, \tau) d\tau \right]
- \psi(\xi, 0) \int_{0}^{t} P_{G}(x, t, \xi, \tau) d\tau \right] \right] \right] d\Gamma(\xi)
\tag{34}
\]

In (32) and (34) the fundamental solution \(G\) is given in Chang et al. [9]. For the case of two-dimensional problems it may be written in the form
\[
G(x, t, \xi, \tau) = \frac{\bar{\sigma}}{4\pi \bar{\zeta}(t - \tau)} \exp \left[ -\frac{C_{\tau}R^2}{4\bar{\zeta}(t - \tau)} \right],
\tag{35}
\]
where \(R\) and \(\bar{\zeta}\) are given by
\[
R^2 = (\bar{\xi}_1 - \bar{x}_1)^2 + (\bar{\xi}_2 - \bar{x}_2)^2
\]
and
\[
\bar{\zeta} = \frac{k_{11} + 2\sigma k_{12} + (\bar{\sigma}^2 + \bar{\sigma})k_{22}}{2}
\]
with
\[
\bar{\xi}_1 = \bar{\xi}_1 + \delta \bar{x}_2, \quad \bar{x}_1 = x_1 + \delta x_2,
\]
\[
\bar{\xi}_2 = \bar{\delta} \bar{x}_2, \quad \bar{x}_2 = \bar{\delta} x_2,
\]
where \(\delta\) and \(\bar{\sigma}\) are, respectively, the real and the positive imaginary parts of the complex root \(\sigma\) of the quadratic
\[
k_{11} + \frac{2k_{12}\sigma}{2} + k_{22}\sigma^2 = 0.
\]

Now from (7), (8) and (10) it follows that the corresponding value of \(P_{\phi}\) is
\[
P_{\phi}(\xi, t) = -P_{\phi}(\xi) \psi(\xi, t) + P_{\phi}(\xi, t) g^{1/2}(\xi),
\tag{36}
\]
where
\[
P_{\phi}(\xi) = \kappa_{\eta} \frac{\partial g^{1/2}}{\partial \bar{\zeta}} n_i.
\tag{37}
\]
Substitution of (10) and (36) in (32) and (34), respectively, gives
\[
\eta(x) \frac{1}{\sqrt{2\pi}} \theta(x, t) = \int_{\Omega} \left[ C_{\tau}g^{1/2}(\xi) \theta(\xi, 0) - H(\xi, g^{1/2}(\xi) \theta(\xi, t)) \right]
+ \int_{\Gamma} \left[ g^{1/2}(\xi) P_{\phi}(\xi, t) \int_{0}^{t} G(x, t, \xi, \tau) d\tau \right]
- \theta(\xi, t) \int_{0}^{t} P_{G}(x, t, \xi, \tau) d\tau \right] \right] \right] d\Gamma(\xi).
\tag{38}
\]

and
\[
\eta(x) \frac{1}{\sqrt{2\pi}} \theta(x, t) = \int_{\Omega} C_{\tau}g^{1/2}(\xi) \theta(\xi, 0) G(x, t, \xi, 0) d\Omega(\xi)
+ \int_{\Gamma} \left[ g^{1/2}(\xi) P_{\phi}(\xi, 0) \int_{0}^{t} G(x, t, \xi, \tau) d\tau \right]
- \theta(\xi, 0) g^{1/2}(\xi) \int_{0}^{t} P_{G}(x, t, \xi, \tau) d\tau \right] \right] \right] d\Gamma(\xi).
\tag{39}
\]

The integral equations (38) and (39) may be used to obtain the solution \(\theta\) of Eq. (5) subject to the initial and boundary conditions (6). Solutions for the temperature \(T\) may then be obtained using (4). Eqs. (38) and (39) are both based on the assumption that for each time step the dependent variable \(\psi\) (and hence the temperature \(T\)) takes on a constant value on the time interval \([0, t]\). In the case of Eq. (38) this constant value is taken to be the value at the upper bound of the semi-open time interval \([0, t]\) and in the case of Eq. (39) the constant value is taken to be the value at the lower bound of the semi-open time interval \([0, t]\).

5. Discretisation and numerical examples

In this section only the integral equation (39) will be considered. It will be used to find numerical solutions to some two-dimensional problems.

Let the domain \(\Omega\) be discretised into \(K\) cells \(\Omega_k\) for \(k = 1, 2, \ldots, K\), the boundary \(\Gamma^0\) into \(J\) elements \(\Gamma_j\) for \(j = 1, 2, \ldots, J\) and the time \(t\) into \(M\) time intervals \([t_{m-1}, t_m]\) for \(m = 1, 2, \ldots, M\). If it is further assumed that the unknowns are constant along any boundary segment \(\Gamma_j = [q_{j-1}, q_j]\) and take on their values at the mid-point \(q_j\), and also
constant in any domain cell \( \Omega_k \) taking on their values at the center point \( r_k \) of the cell. Then the discretised form of (39) for the time element \([t_{m-1}, t_m]\) may be written as

\[
\eta(x)g^{1/2}(x)\Theta(x, t_m) = \sum_{k=1}^{K} \Theta(r_k, t_{m-1}) \int_{\Omega_k} C_r g^{1/2}(\xi) G(x, t_m|\xi, t_{m-1}) \, d\Omega(\xi)
\]

\[
+ \sum_{j=1}^{J} \left[ \int_{I_j} P_\Theta(q_j, t_{m-1}) \int_{t_{m-1}}^{t_m} G(x, t_m|\xi, \tau) d\tau \, d\Gamma(\xi) \right] - \Theta(\Omega_j, t_{m-1}) \left[ \int_{I_j} g^{1/2}(\xi) \int_{t_{m-1}}^{t_m} P_G(x, t_m|\xi, \tau) d\tau \, d\Gamma(\xi) \right] - \int_{I_j} P_g(\xi) \int_{t_{m-1}}^{t_m} G(x, t_m|\xi, \tau) d\tau \, d\Gamma(\xi) \right].
\]

(40)

For each time interval \( m \), (40) is treated as a new problem. It should be noted that the variable \( \Theta \) on the left-hand side of (40) is associated with the time \( t_m \) whereas on the right-hand side it is associated with the time \( t_{m-1} \). Thus in order to solve (40) for the boundary unknowns (\( \Theta \) on \( \Gamma^n \) and \( P_\Theta \) on \( \Gamma^0 \)) using an explicit scheme of linear equations the source point \( x \) has to be located outside the domain \( \Omega \) so that \( \eta = 0 \) and the left-hand side of (40) is zero. Having solved for the boundary unknowns, solutions \( \Theta \) at the interior points \( r_k \) for \( k = 1, 2, \ldots, K \) need to be found. These are then used as initial conditions for the time interval \( m+1 \).

The time integrals of the fundamental solutions \( G \) and \( P_G \) in (40) are evaluated analytically. Specifically

\[
\int_{t_{m-1}}^{t_m} G(x, t_m|\xi, \tau) \, d\tau = \frac{\bar{\sigma}}{4\pi \xi^2} E_1\left(\frac{C_r R^2}{4\xi(t_m-t_{m-1})}\right),
\]

(41)

\[
\int_{t_{m-1}}^{t_m} P_G(x, t_m|\xi, \tau) \, d\tau = \frac{\bar{\sigma}}{4\pi \xi^2 R^2} \exp\left(-\frac{C_r R^2}{4\xi(t_m-t_{m-1})}\right) \frac{\partial R^2}{\partial \xi} \frac{\partial^2 \xi}{\partial \xi^2} n_j,
\]

(42)

where \( C_r \) is assumed to be a positive number and \( E_1 \) is the exponential integral function defined by

\[
E_1(z) = \int_{z}^{\infty} \frac{e^{-s}}{s} \, ds = -\gamma - \ln z + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n!},
\]

(43)

where \( \gamma = 0.57721566490 \) is Euler’s constant.

For all the problems considered the boundary \( \Gamma \) of the domain \( \Omega \) is divided into a number of segments of equal length. Simpson’s \( \frac{1}{3} \) rule is applied for the evaluation of the line integrals. For the area integrals, the domain \( \Omega \) is divided into a number of rectangular cells of equal area. The number of cells is increased to ensure the required level of accuracy for the domain integral. A uniform time step \( \Delta t = t_m - t_{m-1} \) is used for all \( m \). Also, when solving for the boundary unknowns, each source point is located outside the domain at a distance of one and a half times the length of the boundary segment under consideration. The distance is measured from the mid point of each segment in the direction of the outward pointing normal vector \( \mathbf{n} \).

5.1. Problem 1: Quadratic spatial variation of the conductivity matrix

Consider the heat conduction problem governed by (1) for an inhomogeneous anisotropic material for which the coefficients of heat conduction are independent of temperature and are given by Eq. (8). The spatial variation of the coefficients is determined by the function \( g(x) \) which is required to satisfy Eq. (9). Here \( g(x) \) to taken to assume the quadratic form (Fig. 2)

\[
g(x_1^1, x_2^1) = (1 + 0.1x_1^1)^2,
\]

(44)

where \( x_i^1 = x_i/d \) (for \( i = 1, 2 \)) with \( d \) a reference length. Also the coefficients occurring in Eq. (3) take the forms:

\[
\mu_i(x_1^1, x_2^1) = [\kappa_i^1]g(x_1^1, x_2^1) = \left[ \begin{array}{cc} 3 & 1 \\ 1 & 4 \end{array} \right] (1 + 0.1x_1^1)^2,
\]

\[
h(T^1) = 1,
\]

\[
c'(x_1^1, x_2^1, T^1) = \frac{4.5}{T^1} g^{1/2}(x_1^1, x_2^1),
\]

\[
\rho'(x_1^1, x_2^1, T^1) = 1,
\]

where \( \mu_i^1 = \mu_i/\mu_0 \) and \( \kappa_i^1 = \kappa_i/\mu_0 \) with \( \mu_0 \) a reference coefficient of heat conduction, \( T^1 = T/T_0 \) with \( T_0 \) a reference temperature, \( c^1 = c/c_0 \) with \( c_0 \) is a reference heat capacity and \( \rho^1 = \rho/\rho_0 \) with \( \rho_0 = \rho_0 d^2/\rho_0 d^2 \) a reference density.

The geometry, initial and boundary conditions for the test problem under consideration are given in Fig. 3 where

\[
T'(x_1^1, x_2^1, 0) = \frac{1 - 0.25(x_1^1 + x_2^1)^2}{(1 + 0.1x_1^1)^2}
\]

for \( 0 \leq x_1^1 \leq 1 \) and \( 0 \leq x_2^1 \leq 1 \),

\[
T'(x_1^1, 1, t') = \frac{1 - 0.25(1 + x_1^1)^2}{(1 + t')(1 + 0.1x_1^1)^2}
\]

for \( 0 \leq x_1^1 \leq 1 \).

![Fig. 2. Spatial variation in the heat conductivities.](image-url)
for $0 \leq x_1 \leq 1$ and $t' \geq 0$,

$$T'(x_1, 0, t') = 1 - 0.25x_1^2 \quad \text{for} \quad 0 \leq x_1 \leq 1 \quad \text{and} \quad t' \geq 0,$$

$$P'(t, 0, t') = -0.3 - 2.2(1 + x_2') + 0.075(1 + x_2')^2 \quad \text{for} \quad 0 \leq x_2 \leq 1 \quad \text{and} \quad t' \geq 0,$$

where $t' = t/t_0$, $t_0$ is a reference time and $P_T = P_T/P_0$ with $P_0 = k_0T_0/d$ a reference flux.

It may be readily verified using the method of separation of variables that, for this particular problem, Eq. (3) admits the analytical solution

$$T'(x_1, x_2, t) = \frac{1 - 0.25(x_1^2 + x_2^2)}{1 + t'}(1 + 0.1x_1').$$

The corresponding heat flux is given by

$$P'(x_1, x_2, t) = \frac{-0.1(3n_x + n_y)}{1 + t'}[1 - 0.25(x_1^2 + x_2^2)] - \frac{0.5(4n_x + 5n_y)}{1 + t'}(x_1^2 + x_2^2)(1 + 0.1x_1').$$

Since $h(T') = 1$, the variable $\Theta'$ is given by $\Theta' = T'$ where $\Theta = \Theta/T_0$. The function $C(x, \psi)$ in (12) is given by $C'(x, \psi) = 4.5/(g^{1/2}T')$ where $C' = C/C_0$, $C_0 = \rho_0c_0$.

As $0 \leq g^{1/2}T' \leq 1$ it follows that $C'(x, \psi) \geq 4.5$. It is convenient to take the reference value $C_r$ as $C_r = 4.5$ where $C_r = C_r/C_0$. The number of boundary segments and domain cells used is 160 (i.e. $4 \times 40$) and 1600 (i.e. $40 \times 40$), respectively. Table 1 shows a comparison between the BEM and the analytical solutions at the interior point $x = (0.5, 0.5)$ for some different values of the time step $\Delta t$. The results indicate the rate of convergence of the BEM solution to the analytical solution as the time step $\Delta t$ decreases.

### 5.2. Problem 2: Linear temperature variation of the conductivity matrix

Consider the heat conduction problem for a homogeneous isotropic material with coefficients

$$\mu'(x_1', x_2') = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} g(x_1', x_2'),$$

$$h(T') = 1 + T'.$$

$$c'(x_1', x_2', T') = 1 + 0.5T',$$

$$\rho'(x_1', x_2', T') = 1,$$

$$g(x_1', x_2') = 1.$$ 

The initial and boundary conditions are given in Fig. 4.

Since $h(T') = 1 + T'$ (4) gives $\Theta' = T' + \frac{1}{2}T'^2$. Solving for $T'$ and taking the positive square root yields $T' = -1 + \sqrt{2\Theta' + 1}$. The function $C(x, \psi)$ is given as

$$C'(x, \psi) = \frac{1 + 0.5T'}{1 + T'^2}.$$ 

In view of the boundary conditions the temperature $T'$ is expected to satisfy $0 \leq T' \leq 1$, so that $\frac{3}{4} \leq C'(x, \psi) \leq 1$ and hence a reference value of $C_r = 0.75$ is appropriate.

This particular problem is equivalent to the problem in Example 2 of the paper by Goto and Suzuki [7] (for the case when $\kappa = 1$, $\beta = 0.5$). Goto and Suzuki showed that results they obtained using their boundary element technique for $t' = 1$, $x_2' = 0.1$ and values of $x_1'$ given in

![Graph of T'(x_1', x_2', t')](image_url)

**Table 1**

<table>
<thead>
<tr>
<th>Time</th>
<th>BEM solution $T'(0.5, 0.5, t')$ for Problem 1</th>
<th>Analytical solution $T'(0.5, 0.5, t')$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t'$</td>
<td>$\Delta t = 0.001$</td>
<td>$\Delta t = 0.002$</td>
</tr>
<tr>
<td>0.1</td>
<td>0.64907</td>
<td>0.64890</td>
</tr>
<tr>
<td>0.2</td>
<td>0.59494</td>
<td>0.59464</td>
</tr>
<tr>
<td>0.3</td>
<td>0.54914</td>
<td>0.54883</td>
</tr>
<tr>
<td>0.4</td>
<td>0.50989</td>
<td>0.50957</td>
</tr>
<tr>
<td>0.5</td>
<td>0.47587</td>
<td>0.47555</td>
</tr>
<tr>
<td>1.0</td>
<td>0.35685</td>
<td>0.35655</td>
</tr>
</tbody>
</table>
Table 2 agreed with the numerical solution obtained for the same problem using several different numerical techniques. It is therefore reasonable to assume that the figures calculated by them provide a reference solution which is close enough to the exact solution to the problem for the values of \( x_0 \), \( x^2 \), and \( t_0 \) indicated.

By way of comparison the current technique was employed to solve the same problem with the time step decreased and with the boundary elements and corresponding cells increased until convergence was achieved to the values shown in Table 2 using a time step of \( 1/30 \) and 96 boundary elements and 320 (40 \( \times \) 8) cells. The calculated values which are shown in Table 2 agree with the reference solution correct to two decimal places. For convergence to the given values the current explicit method based on the formula (39) requires more boundary elements and cells than the implicit method of Goto and Suzuki which is based on a special case of the formula (38).

### References


