Invertible Ideal and \( v \)-ideal of Ore Extension over a Commutative Dedekind Domain

Amir Kamal Amir

Mathematics Department
Faculty of Mathematics and Natural Sciences
Hasanuddin University
Jl. Perintis Kemerdekaan KM.10, Makassar, Indonesia, 90245.
Email: amirkamalamir@yahoo.com.

Abstract

Let \( R = D[x; \sigma, \delta] \) be an Ore extension over a commutative Dedekind domain \( D \), where \( \sigma \) is an automorphism on \( D \). In this paper we give a relation between invertible ideal in \( D \) with invertible ideal in \( R \). Beside that, we give a sufficient condition for a prime ideal to be a \( v \)-ideal in \( R \).

Keywords: Dedekind, ideal, invertible, \( v \)-ideal
2010 Mathematics Subject Classification: 16S36

1 Introduction

Ore extensions are widely used as the underlying rings of various linear systems investigated in the area algebraic system theory. These systems may represent systems coming from mathematical physics, applied mathematics and engineering sciences which can be described by means of systems of ordinary or partial differential equations, difference equations, differential time-delay equations, etc. If these systems are linear, they can be defined by means of matrices with entries in non-commutative algebras of functional operators such as the ring of differential operators, shift operators, time-delay operators, etc. An important class of such algebras is called Ore extensions (Ore Algebras). This paper studies maximal order and factor rings of an Ore extension over the prime ideals.

2 Definitions and Notations

We recall some definitions, notations, and more or less well known facts concerning.
2.1 Definitions and Notations of v-ideal

In this subsection, let $R$ be an order in a ring $Q$. For any subset $X$ and $Y$ of $Q$, we use the following notations (Marubayashi, Miyamoto, dan Ueda, 1997);

$$(X : Y)_r = \{ q \in Q \mid Yq \subseteq X \}$$

$$(X : Y)_l = \{ q \in Q \mid qY \subseteq X \}$$

and

$$X^{-1} = \{ q \in Q \mid XqX \subseteq X \}.$$

**Definition 2.1** Let $R$ be an order in a ring $Q$. Then a right $R$-submodule $I$ of $Q$ is called a right $R$-ideal of $Q$ if

(i) $I \cap U(Q) \neq \emptyset$ and

(ii) there exists $c \in U(Q)$ such that $cI \subseteq R$.

A left $R$-ideal $I$ of $Q$ is defined similarly. A right and left $R$-ideal is called an $R$-ideal.

**Definition 2.2** For any right $R$-ideal $I$, set $I_v = (R : (R : I)_l)_r$ and we call $I_v$ a right $v$-ideal if $I_v = I$. Similarly, we set $vJ = (R : (R : J)_r)_l$ for any left $R$-ideal $J$ and $J$ is said to be a left $v$-ideal if $vJ = J$. An $R$-ideal $I$ is called a $v$-ideal if $I_v = I = vI$ and An $R$-ideal $A$ is called invertible if

$$(R : A)_lA = R = A(R : A)_r.$$

2.2 Definitions and Notations of Ore Extension

A (left) skew derivation on a ring $D$ is a pair $(\sigma, \delta)$ where $\sigma$ is a ring endomorphism of $D$ and $\delta$ is a (left) $\sigma$-derivation on $D$; that is, an additive map from $D$ to itself such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in D$. For $(\sigma, \delta)$ any skew derivation on a ring $D$, we obtain

$$\delta(a^m) = \sum_{i=0}^{m-1} \sigma(a)^i\delta(a)a^{m-1-i}$$

for all $a \in D$ and $m = 1, 2, \cdots$ (See [2, Lemma 1.1])

**Definition 2.3** Let $D$ be a ring with identity 1 and $(\sigma, \delta)$ be a (left) skew derivation on the ring $D$. The Ore Extension over $D$ with respect to the skew derivation $(\sigma, \delta)$ is the ring consisting of all polynomials over $D$ with an indeterminate $x$ denoted by:

$$D[x; \sigma, \delta] = \{ f(x) = a_nx^n + \cdots + a_0 \mid a_i \in D \}$$

satisfying the following equation, for all $a \in D$

$$xa = \sigma(a)x + \delta(a).$$

The notations $D[x; \sigma]$ stand for the particular Ore extensions where $\delta = 0$ and $D[x; \delta]$ for $\sigma$ the identity map. In this paper, we describe the Ore Extension $R = D[x; \sigma, \delta]$ where $D$ is a commutative Dedekind domain and $\sigma$ is an automorphism.
The Ore extension $D[x; \sigma, \delta]$ is a free left $D$-module with basis $1, x, x^2, \cdots$. To abbreviate the assertion, the symbol $R$ stands for the Ore extension $D[x; \sigma, \delta]$ constructed from a ring $D$ and a skew derivation $(\sigma, \delta)$ on $D$. The degree of a nonzero element $f \in R$ is defined in the obvious fashion. Since the standard form for elements of $R$ is with left-hand coefficients, the leading coefficient of $f$ is $f_n$ if $f(x) = f_0 + f_1 x + \cdots + f_{n-1} x^{n-1} + f_n x^n$ with all $f_i \in D$ and $f_n \neq 0$. If $\sigma$ is an automorphism, $f$ can also be written with right-hand coefficients, but then its $x^n$-coefficient is $\sigma^{-n}(f_n)$. While a general formula for $x^n a$ where $a \in D$ and $n \in \mathbb{N}$ is too involved to be of much use, an easy induction establishes that

$$x^n a = \delta^n(a) + a_1 x + \cdots + a_{n-1} x^{n-1} + \sigma^n(a)x^n$$

for some $a_1, \cdots, a_{n-1} \in D$.

In preparation for our analysis of the types of ideals occurred when prime ideals of an Ore extension $D[x; \sigma, \delta]$ are contracted to the coefficient ring $D$, we consider $\sigma$-prime, $\delta$-prime, and $(\sigma, \delta)$-prime ideals of $D$.

**Definition 2.4** Let $\Sigma$ be a set of maps from the ring $D$ to itself. A $\Sigma$-ideal of $D$ is any ideal $I$ of $D$ such that $\alpha(I) \subseteq I$ for all $\alpha \in \Sigma$. A $\Sigma$-prime ideal is any proper $\Sigma$-ideal $I$ such that whenever $J, K$ are $\Sigma$-ideals satisfying $JK \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$.

In the context of a ring $D$ equipped with a skew derivation $(\sigma, \delta)$, we shall make use of the above definition in the cases $\Sigma = \{\sigma\}, \Sigma = \{\delta\}$ and $\Sigma = \{\sigma, \delta\}$; and simplify the prefix $\Sigma$ to respectively $\sigma$, $\delta$, or $(\sigma, \delta)$.

## 3 Invertible Ideal and $v$-ideal of Skew Polynomial Ring

Throughout this section, let $D$ be a commutative Dedekind domain with quotient ring $K$ and $R = D[x; \sigma, \delta]$ be the Ore extension over $D$, for $(\sigma, \delta)$ is a skew derivation, $\sigma \neq 1$ is an automorphism of $D$ and $\delta \neq 0$.

We give a relation between invertible ideal in $D$ with invertible ideal in $R$.

**Theorem 3.1** Let $p[x, \sigma, \delta]$ be an ideal of $R$ where $p$ is a $(\sigma, \delta)$-ideal of $D$. Then $p[x, \sigma, \delta]$ is an invertible ideal if $p$ is an invertible ideal.

**Proof.** Let $p$ be an invertible ideal, then $(D : p)_l p = D = p(D : p)_r$ according to Definition 2.2. Using again Definition 2.2 to prove the lemma, it will be shown that

$$\left( R : p[x, \sigma, \delta] \right)_l p[x, \sigma, \delta] = R = p[x, \sigma, \delta] \left( R : p[x, \sigma, \delta] \right)_r.$$
It is easy to see that
\[
(R : p[x, \sigma, \delta])_l p[x, \sigma, \delta] \subseteq R.
\]
On the other side, let \( f(x) = f_n x^n + \cdots + f_0 \in R \). Since \((D : p)_l p = D = p(D : p)_r\) dan \( f_i \in D \) for all \( i \), then
\[
f_n = q_{n1}p_{n1} + \cdots + q_{nm}p_{nm}, \quad \cdots, \quad f_0 = q_{01}p_{01} + \cdots + q_{0m}p_{0m}
\]
where \( q_{jk} \in (D : p)_l \), dan \( p_{jk} \in p \). So,
\[
f(x) = \left[q_{n1}p_{n1}x^n + \cdots + q_{01}p_{01}\right] + \cdots + \left[q_{nm}p_{nm}x^n + \cdots + q_{0m}p_{0m}\right].
\]
Since \( q_{jk} \in (D : p)_l \), then \( q_{jk} \in (R : p[x, \sigma, \delta])_l \). This shows that, \( f(x) \in (R : p[x, \sigma, \delta])_l p[x, \sigma, \delta] \). This implies, \( R \subseteq (R : p[x, \sigma, \delta])_l p[x, \sigma, \delta] \). Therefore
\[
(R : p[x, \sigma, \delta])_l p[x, \sigma, \delta] = R.
\]
With similar way, it can be shown
\[
p[x, \sigma, \delta] (R : p[x, \sigma, \delta])_r = R.
\]
This completes the proof. \( \blacksquare \).

Now, we give sufficient condition for a prime ideal to be a \( \nu \)-ideal in \( R \). We start with to lemmas.

**Lemma 3.1** Let \( P \) be a prime ideal in \( R \). Then
\[
(K[x, \sigma, \delta] : K[x, \sigma, \delta](R : P)_l)_r = (K[x, \sigma, \delta] : (K[x, \sigma, \delta] : PK[x, \sigma, \delta])_l)_r
\]
Proof. Let \( k_1q_1 + \cdots + k_nq_n \in K[x, \sigma, \delta](R : P)_l \) where \( k_i \in K[x, \sigma, \delta] \) and \( q_i \in (R : P)_l \) for all \( i \), then \( q_i P \subseteq R \). This implies
\[
(k_1q_1 + \cdots + k_nq_n)PK[x, \sigma, \delta] \subseteq K[x, \sigma, \delta].
\]
This shows that \( k_1q_1 + \cdots + k_nq_n \in (K[x, \sigma, \delta] : PK[x, \sigma, \delta])_l \). Therefore
\[
K[x, \sigma, \delta](R : P)_l \subseteq (K[x, \sigma, \delta] : PK[x, \sigma, \delta])_l.
\]
Now, it is easy to see that
\[
(K[x, \sigma, \delta] : K[x, \sigma, \delta](R : P)_l)_r \supseteq (K[x, \sigma, \delta] : (K[x, \sigma, \delta] : PK[x, \sigma, \delta])_l)_r.
\]
Conversely, let \( q \in (K[x, \sigma, \delta] : K[x, \sigma, \delta](R : P)_l) \). This means \( q \in Q \) dan \( K[x, \sigma, \delta](R : P)_l \). Moreover, let \( t \in (K[x, \sigma, \delta] : PK[x, \sigma, \delta])_l \). This means \( t \in Q \) dan \( tPK[x, \sigma, \delta] \subseteq K[x, \sigma, \delta] \). So, \( tP \subseteq K[x, \sigma, \delta] \).
Since \( R = \mathbf{D}[x, \sigma, \delta] \) where \( D \) is a Dedekind domain, then \( R \) is Noetherian by McConnell
and Robson [4]. So, since $P$ is an ideal in Noetherian ring $R$, then $P$ is finitely generated. Let $P = \langle p_1, \ldots, p_n \rangle \text{ and } t_p_i = k_i$ where $k_i \in K[x, \sigma, \delta]$ for all $i = 1, \ldots, n$. So, we can get a $d \in D$ such that $dk_i \in R$ or $d(t_p_i) = dk_i \in R$ for all $i$. This implies $(dt)p \in R$ for all $p \in P$ or $dt \in (R : P)_l$. Using the fact that $K[x, \sigma, \delta](R : P)_l q \subseteq K[x, \sigma, \delta]$, we get $tq = [d^{-1}][dt]q \in K[x, \sigma, \delta]$. This means

$$q \in \left(K[x, \sigma, \delta] : (K[x, \sigma, \delta] : PK[x, \sigma, \delta])_l\right)_r.$$ 

So,

$$\left(K[x, \sigma, \delta] : K[x, \sigma, \delta](R : P)_l\right)_r \subseteq \left(K[x, \sigma, \delta] : (K[x, \sigma, \delta] : PK[x, \sigma, \delta])_l\right)_r.$$ 

This relation completes the proof. ■.

**Lemma 3.2** Let $P$ be a prime ideal in $R$. Then

$$\left(K[x, \sigma, \delta] : K[x, \sigma, \delta](R : P)_l\right)_r = \left(R : (R : P)_l\right)_r K[x, \sigma, \delta].$$

**Proof.**

Step I.

Let $q \in \left(K[x, \sigma, \delta] : K[x, \sigma, \delta](R : P)_l\right)_r$, then

$$K[x, \sigma, \delta](R : P)_l q \subseteq K[x, \sigma, \delta].$$

Since $(R : P)_l$ is a $R$-submodul of Noether ring $Q$, then $(R : P)_l$ is finitely genereted. Let $(R : P)_l = Rq_1 + \cdots + Rq_n$ where $q_i \in (R : P)_l$ for all $i$. So, from (1) we get

$$\left(R : (R : P)_l\right)_r q = (Rq_1 + \cdots + Rq_n)q \subseteq K[x, \sigma, \delta].$$

It easy to see that $q_i q = k_i$ for some $k_i \in K[x, \sigma, \delta]$ for all $i$. So, for all $i$ there is a $d_i \in D$ where $(q_i d_i) = k_i d_i \in R$. Therefore, there is a $d \in D$ such that

$$\left(R : (R : P)_l\right)_r (qd) = (Rq_1 + \cdots + Rq_n)(qd) \subseteq R.$$

This shows that $qd \in \left(R : (R : P)_l\right)_r$. So, $q = (qd)(d^{-1}) \in \left(R : (R : P)_l\right)_r K[x, \sigma, \delta]$, since $qd \in \left(R : : (R : P)_l\right)_r$ and $d^{-1} \in K[x, \sigma, \delta]$. Therefore we have

$$\left(K[x, \sigma, \delta] : K[x, \sigma, \delta](R : P)_l\right)_r \subseteq \left(R : (R : P)_l\right)_r K[x, \sigma, \delta].$$

Step II.

Let

$$q = q_1 k_1 + \cdots + q_n k_n \in \left(R : (R : P)_l\right)_r K[x, \sigma, \delta]$$

where
where \( q_i \in \left( R : (R : P)_i \right)_r \) dan \( k_i \in K[x, \sigma, \delta] \) for all \( i \). This means \( (R : P)_i q_i \subseteq R \) and \( K[x, \sigma, \delta] (R : P)_i q_i \subseteq K[x, \sigma, \delta] \). This implies \( K[x, \sigma, \delta] (R : P)_i q_i k_i \subseteq K[x, \sigma, \delta] \) for all \( i \). Therefore \( K[x, \sigma, \delta] (R : P)_i (q_1 k_1 + \cdots + q_n k_n) \subseteq K[x, \sigma, \delta] \) or

\[
q \in \left( K[x, \sigma, \delta] : K[x, \sigma, \delta] (R : P)_i \right)_r.
\]

This relation implies

\[
\left( K[x, \sigma, \delta] : K[x, \sigma, \delta] (R : P)_i \right)_r \supseteq \left( R (R : P)_i \right)_r K[x, \sigma, \delta]
\]

Form step I and step II, the theorem follows. ■.

4 Acknowledgement

The author is very grateful to Prof. Pudji Astuti, Dr. Intan Muchtadi-Alamsyah, Prof. Irawati, and Prof. Hidetoshi Marubayashi for various discussions on studying properties of Ore extensions.

References


