Real matrix representation for trinion numbers and its properties

Mawardi Bahri\textsuperscript{a}, Mulkiah\textsuperscript{a}, Nur Erawati\textsuperscript{a} and Ryuichi Ashino\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Hasanuddin University, Makassar 90245, Indonesia 
\hspace{1cm} e-mail: mawardibahri@gmail.com
\textsuperscript{b} Division of Mathematical Sciences, Osaka Kyouiku University, Osaka 582-8582, Japan 
\hspace{1cm} e-mail: ashino@cc.osaka-kyoiku.ac.jp

Abstract

In this paper, we show that the trinion numbers can be explicitly represented using real matrices. Some important properties of real matrix representations for the trinion are investigated in detail.

Keywords: trinion numbers, real matrix representation

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1 Introduction

As we know, real quaternions were first introduced by Sir W. R. Hamilton in 1843 as one of generalizations of the complex numbers to higher spatial dimensions (see, for example, \cite{2,4}). Denote by $\mathbb{R}$, the set of real numbers. The set of real quaternion can be expressed as

$$\mathbb{H} = \{ q = q_0 + i q_1 + j q_2 + k q_3 : q_0, q_1, q_2, q_3 \in \mathbb{R} \},$$

where

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad ik = -ki, \quad jk = -kj, \quad \text{and} \quad ijk = -1.$$  

Unlike multiplication of complex numbers, multiplication of quaternions is non commutative. So, it is not easy to extend various results on similarity complex numbers to quaternions. It is well known that real quaternion is isomorphic to a $4 \times 4$ real matrix and quaternion matrices can be represented using complex matrices. However, due to the noncommutativity of quaternion, many properties of complex matrices cannot be established for quaternion matrices. For some recent work and surveys on the subject, we refer the reader to \cite{3,5,7,8,9,10,11} and the references therein. On the other hand, the trinion number is also a generalization of the complex number to higher spatial dimensions, which is closely related to quaternion numbers. So far as we know, the relationship between trinion and its real matrix representation is still unknown.

Therefore, in the present work, we first show that a trinion number is algebraically isomorphic to a $3 \times 3$ real matrix. Then, we investigate some important properties of real matrix representations for the trinion numbers and obtain their corresponding properties of real matrix representations for the quaternion matrices, which have been studied in \cite{6}.
2 Trinion Numbers

In this section, we provide some basic facts about trinions. For the necessary background details, we refer the reader [1] and the references therein. The trinion numbers can be regarded as a generalization of the complex algebra. Because a trinion is defined with one real and two imaginary components. Then, the set of trinion numbers \( T \) consists of elements of the form:

\[
T = \{ t = t_0 + it_1 + jt_2 : t_0, t_1, t_2 \in \mathbb{R} \},
\]

where elements \( i \) and \( j \) satisfy the following multiplication rules:

\[
ij = ji = -1, \quad i^2 = j, \quad j^2 = -i.
\]

We may express a trinion as scalar part, denoted by \( \text{Sc}(t) = t_0 \), and a pure trinion, denoted by \( \text{Vec}(t) = it_1 + jt_2 = t \). According to (4), it is easily seen that the trinion multiplication is commutative. It also is associative and distributive with respect to both addition and multiplication. Therefore, the multiplication of any trinions

\[
t_1 = p_0 + ip_1 + jp_2
\]

and

\[
t_2 = q_0 + iq_1 + jq_2,
\]

is obtained by

\[
t_1t_2 = t_2t_1 = (p_0 + ip_1 + jp_2)(q_0 + iq_1 + jq_2)
\]

\[
= p_0q_0 + ip_0q_1 + jp_0q_2 + ip_1q_0 + i^2p_1q_1 + ip_1q_2 + jp_2q_0 + ip_2q_1 + j^2p_2q_2
\]

\[
= p_0q_0 + ip_0q_1 + jp_0q_2 + ip_1q_0 + jp_1q_1 - p_1q_2 + jp_2q_0 - p_2q_1 - ip_2q_2
\]

\[
= (p_0q_0 - p_1q_2 - p_2q_1) + i(p_0q_1 + p_1q_0 - p_2q_2) + j(p_0q_2 + p_1q_1 + p_2q_0). \tag{7}
\]

Following [1], we define the conjugate of trinion \( t \) by

\[
\bar{t} = \frac{(t_0^2 + t_1^2 + t_2^2)(t_0^3 + t_1t_2 - t_0^2t_1) - j(t_0t_2 - t_1^2)}{t_0^3 - t_0^2 + t_0^1 + 3t_0^1t_2}
\]

\[
= \frac{(t_0^2 + t_1^2 + t_2^2)(t_0^3 + t_1t_2 - i(t_2^2 + t_0t_1) - j(t_0t_2 - t_1^2))}{(t_0 + t_2 - t_1)(t_0^3 + t_1^2 + t_2^2 + t_0^1 + t_1^1 + t_2^1 - t_0^2t_1)}. \tag{8}
\]

The norm or modulus of a trinion can be written as

\[
|t| = \sqrt{\bar{t}t} = \sqrt{t_0^2 + t_1^2 + t_2^2}. \tag{9}
\]

3 Matrix Representation of Trinion

For each trinion \( t = t_0 + it_1 + jt_2 \in T \), define a real matrix \( M_t \in \mathbb{R}^{3 \times 3} \) by

\[
M_t = \begin{pmatrix}
t_0 & t_1 & t_2 \\
-t_2 & t_0 & t_1 \\
-t_1 & -t_2 & t_0
\end{pmatrix},
\]
and define a mapping \( f \) by \( f : t \mapsto M_t \). Denote by \( M_3(\mathbb{R}) = \{ M_t : t \in \mathbb{T} \} \), the image of \( \mathbb{T} \) by the mapping \( f \).

Then, it follows that

\[
i = 0 + i + 0 \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad j = 0 + 0 + j \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},
\]

\[
I = 1 + 0 + 0 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

(10)

which are called the base matrices of matrix representations of trinion. Next, note that

\[
\det M_t = t_0 \det \begin{pmatrix} t_0 & t_1 \\ -t_2 & t_0 \end{pmatrix} - t_1 \det \begin{pmatrix} -t_2 & t_1 \\ -t_1 & t_0 \end{pmatrix} + t_2 \det \begin{pmatrix} -t_2 & t_0 \\ -t_1 & t_2 \end{pmatrix}
= 3t_0t_1t_2 + t_0^3 - t_1^3 + t_2^3,
\]

and the transpose of \( f(t) \), denoted by \( (f(t))^t \), is given by

\[
(f(t))^t = \begin{pmatrix} t_0 & -t_2 & -t_1 \\ t_1 & t_0 & -t_2 \\ t_2 & t_1 & t_0 \end{pmatrix}.
\]

(11)

(12)

**Theorem 3.1.** The mapping \( f : \mathbb{T} \rightarrow M_3(\mathbb{R}) \) is a homomorphism.

**Proof.** It is enough to show that \( f(\cdot) \) satisfies the following three properties:

(i) \( f(u + v) = f(u) + f(v) \), \( u, v \in \mathbb{T} \),

(ii) \( f(\omega) = f(u)f(v) \), \( u, v \in \mathbb{T} \),

(iii) \( f(\alpha u) = \alpha f(u) \), \( \alpha \in \mathbb{R}, u \in \mathbb{T} \).

Let us show (i) and (ii) only, because (iii) is trivial. For (i), we have

\[
f(u + v) = f((u_0 + iu_1 + ju_2) + (v_0 + iv_1 + jv_2))
= f((u_0 + v_0) + i(u_1 + v_1) + j(u_2 + v_2))
= \begin{pmatrix} u_0 + v_0 & u_1 + v_1 & u_2 + v_2 \\ -(u_2 + v_2) & u_0 + v_0 & u_1 + v_1 \\ -(u_1 + v_1) & -(u_2 + v_2) & u_0 + v_0 \end{pmatrix}
= \begin{pmatrix} v_0 & v_1 & v_2 \\ -v_2 & v_0 & v_1 \\ -v_1 & v_2 & v_0 \end{pmatrix}
= f(u) + f(v).
\]
For (ii), we have
\[ f(\mathbf{uv}) = f((u_0 + iu_1 + jv_2)(v_0 + iv_1 + jv_2)) = f((u_0v_0 - u_1v_2 - u_2v_1 + i(u_0v_1 + u_1v_0 - u_2v_2) + j(u_0v_2 + u_1v_1 + u_2v_0)) = (u_0v_0 - u_1v_2 - u_2v_1, u_0v_1 + u_1v_0 - u_2v_2, u_0v_2 + u_1v_1 + u_2v_0) \]
\[ = f(u_0v_0 - u_1v_2 - u_2v_1 + u_0v_1 + u_1v_0 - u_2v_2, u_0v_1 + u_1v_0 - u_2v_2) = (u_0v_0 - u_1v_2 - u_2v_1, u_0v_1 + u_1v_0 - u_2v_2) \]
\[ = f(u_0 + iu_1 + jv_2)f(v_0 + iv_1 + jv_2) = f(u)f(v), \]
which proves the theorem. \( \square \)

Note that the mapping \( f : T \rightarrow M_3(\mathbb{R}) \) is a monomorphism, because \( f(u) \neq f(v) \) implies
\[ \begin{pmatrix} u_0 & u_1 & u_2 \\ -u_2 & u_0 & u_1 \\ -u_1 & -u_2 & u_0 \end{pmatrix} \neq \begin{pmatrix} v_0 & v_1 & v_2 \\ -v_2 & v_0 & v_1 \\ -v_1 & -v_2 & v_0 \end{pmatrix}. \] (14)

4 Main Results

**Theorem 4.1.** For \( t \in T \), let \( \overrightarrow{t} = \begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} \) be the vector representation of the trinion. Then, for every \( u = u_0 + iu_1 + jv_2 \in T \), we obtain
\[ \overrightarrow{ut} = (f(u))^{T}\overrightarrow{t}. \] (15)

**Proof.** We have
\[ (f(u))^{T}\overrightarrow{t} = \begin{pmatrix} u_0 & u_1 & u_2 \\ -u_2 & u_0 & u_1 \\ -u_1 & -u_2 & u_0 \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} u_0t_0 - u_2t_1 - u_1t_2 \\ u_1t_0 + u_0t_1 - u_2t_2 \\ u_2t_0 + u_1t_1 + u_0t_2 \end{pmatrix}. \] (16)

On the other hand,
\[ ut = (u_0t_0 - u_2t_1 - u_1t_2) + i(u_1t_0 + u_0t_1 - u_2t_2) + j(u_2t_0 + u_1t_1 + u_0t_2), \] (17)
which completes the proof. \( \square \)

**Theorem 4.2.** Let \( M = \begin{pmatrix} 1 \\ i \\ j \end{pmatrix} \). If \( u = u_0 + iu_1 + jv_2 \in T \), then \( f(u)M = Mu \).
Proof. Indeed, we have
\[
f(u)M = \begin{pmatrix} u_0 & u_1 & u_2 \\ -u_2 & u_0 & u_1 \\ -u_1 & -u_2 & u_0 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ j \end{pmatrix} = \begin{pmatrix} u_0 + iu_1 + ju_2 \\ -u_2 + iu_0 + ju_1 \\ -u_1 - iu_2 + ju_0 \end{pmatrix}
\]
which completes the proof. \(\Box\)

Theorem 4.3. Let
\[
N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} = f(i), \quad N_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = f(j).
\]
For every matrix \(M\), the following relations are fulfilled:
\[
N_1 M = M N_1, \quad N_2 M = M N_2.
\]
Proof. It directly follows from the multiplication rules of the trinion numbers that
\[
N_1 M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ j \end{pmatrix} = \begin{pmatrix} 1 \\ i \\ j \end{pmatrix} = M = N_1 M,
\]
and
\[
N_2 M = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ j \end{pmatrix} = \begin{pmatrix} 1 \\ i \\ j \end{pmatrix} = M = N_2 M,
\]
which completes the proof. \(\Box\)

Theorem 4.4. Let \(t \in \mathbb{T}, t = a + bi + cj\). Then
\[
f(t^2) = f^2(t).
\]
Proof. Using (4), we have
\[
t^2 = (a^2 - 2bc) + i(2ab - c^2) + j(b^2 + 2ac).
\]
This means that
\[
f(t^2) = \begin{pmatrix} a^2 - 2bc & 2abc - c^2 & b^2 + 2ac \\ -(b^2 + 2ac) & a^2 - 2bc & 2abc - c^2 \\ -(2abc - c^2) & -(b^2 + 2ac) & a^2 - 2bc \end{pmatrix}.
\]
Hence,

\[ f^2(t) = f(t)f(t) \]

\[ = \begin{pmatrix} a & b & c \\ -c & a & b \\ -b & -c & a \end{pmatrix} \begin{pmatrix} a & b & c \\ -c & a & b \\ -b & -c & a \end{pmatrix} \]

\[ = \begin{pmatrix} a^2 - bc - b^2 & ab + ab - c^2 & ac + b^2 + ca \\ -ca - ac - b^2 & -cb + a^2 - bc & -c^2 + ab + ba \\ -ba + c^2 - ab & -b^2 - ac - ac & -bc - cb + a^2 \end{pmatrix} \]

\[ = \begin{pmatrix} a^2 - 2bc & 2abc - c^2 & b^2 + 2ac \\ -(b^2 + 2ac) & a^2 - 2bc & 2abc - c^2 \\ -(2abc - c^2) & -(b^2 + 2ac) & a^2 - 2bc \end{pmatrix} \]

\[ = f(t^2), \]

which completes the proof.

**Theorem 4.5.** Let \( t = a + bi + cj \in T \). Then, we have

\[ 2(Sc(t))f(t) - f^2(t) + f^2(t) = f^2(Sc(t)). \] (25)

**Proof.** Firstly we have

\[ f^2(t) = \begin{pmatrix} a^2 - 2bc & 2abc - c^2 & b^2 + 2ac \\ -(b^2 + 2ac) & a^2 - 2bc & 2abc - c^2 \\ -(2abc - c^2) & -(b^2 + 2ac) & a^2 - 2bc \end{pmatrix}, \] (26)

\[ f^2(t) = \begin{pmatrix} -2bc & -c^2 & b^2 \\ -b^2 & -2bc & -c^2 \\ c^2 & -b^2 & -2bc \end{pmatrix}. \] (27)

Then, we have

\[ 2(Sc(t))f(t) - f^2(t) + f^2(t) \]

\[ = 2a \begin{pmatrix} a & b & c \\ -c & a & b \\ -b & -c & a \end{pmatrix} - \begin{pmatrix} a^2 - 2bc & 2abc - c^2 & b^2 + 2ac \\ -(b^2 + 2ac) & a^2 - 2bc & 2abc - c^2 \\ -(2abc - c^2) & -(b^2 + 2ac) & a^2 - 2bc \end{pmatrix} \]

\[ + \begin{pmatrix} -2bc & -c^2 & b^2 \\ -b^2 & -2bc & -c^2 \\ c^2 & -b^2 & -2bc \end{pmatrix} \]

\[ = \begin{pmatrix} 2a^2 & 2ab & 2ac \\ -2ac & 2a^2 & 2ab \\ -2ab & 2ac & 2a^2 \end{pmatrix} - \begin{pmatrix} a^2 - 2bc & 2abc - c^2 & b^2 + 2ac \\ -(b^2 + 2ac) & a^2 - 2bc & 2abc - c^2 \\ -(2abc - c^2) & -(b^2 + 2ac) & a^2 - 2bc \end{pmatrix} \]

\[ + \begin{pmatrix} -2bc & -c^2 & b^2 \\ -b^2 & -2bc & -c^2 \\ c^2 & -b^2 & -2bc \end{pmatrix} \]

\[ = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix} \]
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\[
\begin{pmatrix}
  a & 0 & 0 \\
  0 & a & 0 \\
  0 & 0 & a \\
\end{pmatrix}
\begin{pmatrix}
  a & 0 & 0 \\
  0 & a & 0 \\
  0 & 0 & a \\
\end{pmatrix}
= f(\text{Sc}(t))f(\text{Sc}(t)) = f^2(\text{Sc}(t)).
\]

Theorem 4.6. Let \( t = a + bi + cj \in \mathbb{T} \). Then, we have

\[
f^2(t) - 2(\text{Sc}(t))f(t) + f(t)(f(t))^\dagger = 0. \tag{28}
\]

Proof. It is not difficult to check that

\[
f^2(t) = \begin{pmatrix}
  a^2 - 2bc & 2abc - c^2 & b^2 + 2ac \\
  -(b^2 + 2ac) & a^2 - 2bc & 2abc - c^2 \\
  -(2abc - c^3) & -(b^2 + 2ac) & a^2 - 2bc
\end{pmatrix}, \tag{29}
\]

and

\[
f(t)(f(t))^\dagger = \begin{pmatrix}
  a^2 + 2bc & c^2 & -b^2 \\
  b^2 & a^2 + 2bc & c^2 \\
  -c^2 & b^2 & a^2 + 2bc
\end{pmatrix}. \tag{30}
\]

Then, we have

\[
f^2(t) - 2(\text{Sc}(t))f(t) + f(t)(f(t))^\dagger
= \begin{pmatrix}
  a^2 - 2bc & 2abc - c^2 & b^2 + 2ac \\
  -(b^2 + 2ac) & a^2 - 2bc & 2abc - c^2 \\
  -(2abc - c^3) & -(b^2 + 2ac) & a^2 - 2bc
\end{pmatrix} - 2a \begin{pmatrix}
  a & b & c \\
  -c & a & b \\
  -b & -c & a
\end{pmatrix}
+ \begin{pmatrix}
  a^2 + 2bc & c^2 & -b^2 \\
  b^2 & a^2 + 2bc & c^2 \\
  -c^2 & b^2 & a^2 + 2bc
\end{pmatrix}
= \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix},
\]

which proves the theorem. \(\square\)

Theorem 4.7. Let \( t = a + bi + cj \in \mathbb{T} \). Then, we have

\[
f(t^{-1}) = f^{-1}(t). \tag{31}
\]

Proof. From the inverse of trinion, we immediately obtain

\[
t^{-1} = \frac{(a^2 + bc - i(c^2 + ab) - j(ac - b^2))}{(a^3 - b^3 + c^3 + 3abc)}. \tag{32}
\]
Then, we have.

\[ f(t^{-1}) = \begin{pmatrix}
\frac{a^2+bc}{(a^3-b^3+c^3+3abc)} & \frac{-(a^2+bc)}{(a^3-b^3+c^3+3abc)} & \frac{-(ac-b^2)}{(a^3-b^3+c^3+3abc)} \\
\frac{-(a^2+b^3+c^3+3abc)}{(a^3-b^3+c^3+3abc)} & \frac{a^2+bc}{(a^3-b^3+c^3+3abc)} & -\frac{bc}{(a^3-b^3+c^3+3abc)} \\
\frac{bc}{(a^3-b^3+c^3+3abc)} & \frac{a^2+bc}{(a^3-b^3+c^3+3abc)} & -\frac{ac-b^2}{(a^3-b^3+c^3+3abc)}
\end{pmatrix}. \quad (33)

It is known that

\[ f^{-1}(t) = \begin{pmatrix}
 a & b & c \\
-(-c) & a & b \\
-(-b) & -c & a
\end{pmatrix}^{-1}. \quad (34)

Long computations imply

\[ f^{-1}(t) = \begin{pmatrix}
\frac{a^2+bc}{(a^3-b^3+c^3+3abc)} & \frac{-(a^2+bc)}{(a^3-b^3+c^3+3abc)} & \frac{-(ac-b^2)}{(a^3-b^3+c^3+3abc)} \\
\frac{-(a^2+b^3+c^3+3abc)}{(a^3-b^3+c^3+3abc)} & \frac{a^2+bc}{(a^3-b^3+c^3+3abc)} & \frac{bc}{(a^3-b^3+c^3+3abc)} \\
\frac{bc}{(a^3-b^3+c^3+3abc)} & \frac{a^2+bc}{(a^3-b^3+c^3+3abc)} & \frac{ac-b^2}{(a^3-b^3+c^3+3abc)}
\end{pmatrix}, \quad (35)

which completes the proof.

**Theorem 4.8.** Let \( p = p_0 + ip_1 + jp_2, q = q_0 + iq_1 + jq_2 \in \mathbb{T}. \) Then, we have

\[ f(pq) = f(p)f(q). \quad (36) \]

**Proof.** Applying (7) and (8), we easily get

\[
\begin{align*}
\bar{p} &= \frac{(p_0^2 + p_1^2 + p_2^2)(p_0^2 + p_1p_2 - iq_0^2 + p_0p_1) - j(p_0p_2 - p_1^2)}{(p_0 + p_2 - p_1)(p_0^2 + p_1^2 + p_2^2 + p_0p_1 + p_1p_2 - p_0p_2)}, \\
\bar{q} &= \frac{(q_0^2 + q_1^2 + q_2^2)(q_0^2 + q_1q_2 - iq_0^2 - iq_1q_0 - j(q_0q_2 - q_1^2))}{(q_0 + q_2 - q_1)(q_0^2 + q_1^2 + q_2^2 + q_0q_1 + q_1q_2 - q_2q_0)}, \\
\bar{pq} &= \frac{(p_0^2 + p_1^2 + p_2^2)(q_0^2 + q_1^2 + q_2^2)(A - Bi + Cj)}{(p_0^2 - p_1^2 + p_2^2 + 3p_0p_1p_2)(q_0^2 - q_1^2 + q_2^2 + 3q_0q_1q_2)},
\end{align*}
\]

where

\[
\begin{align*}
A &= (p_1p_2 + p_0^2)(q_1q_2 + q_0^2) - (p_2^2 + p_1p_0)(q_0q_2 - q_1^2) - (p_0p_2 - p_1^2)(q_0^2 + q_1q_0), \\
B &= -(p_1p_2 + p_0^2)(q_2 + q_1q_0) - (p_2^2 + p_1p_0)(q_1q_2 + q_0^2) - (p_0p_2 - p_1^2)(q_0q_2 - q_1^2), \\
C &= -(p_1p_2 + p_0^2)(q_0q_2 - q_1^2) + (p_2^2 + p_1p_0)(q_0q_2 - q_1^2) - (p_0p_2 - p_1^2)(q_1q_2 + q_0^2).
\end{align*}
\]

It means that

\[
\begin{align*}
f(pq) = \frac{(p_0^2 + p_1^2 + p_2^2)(q_0^2 + q_1^2 + q_2^2)}{(p_0^2 - p_1^2 + p_2^2 + 3p_0p_1p_2)(q_0^2 - q_1^2 + q_2^2 + 3q_0q_1q_2)} \begin{pmatrix}
A & B & C \\
-C & A & B \\
-B & -C & A
\end{pmatrix}.
\]

(37)
Then, we have

\[
\begin{align*}
  f(p) &= \frac{p_0^2 + p_1^2 + p_2^2}{p_0^2 - p_1^2 + p_2^2 + 3p_0p_1p_2} \quad \begin{pmatrix}
  p_1p_2 + p_0^2 & -(p_2^2 + p_0p_1) & -(p_0p_2 - p_1^2) \\
  (p_0p_2 - p_1^2) & (p_1p_2 + p_0^2) & -(p_2^2 + p_0p_1) \\
  (p_2^2 + p_0p_1) & (p_0p_2 - p_1^2) & (p_1p_2 + p_0^2)
\end{pmatrix} \\
  f(q) &= \frac{q_0^2 + q_1^2 + q_2^2}{q_0^2 - q_1^2 + q_2^2 + 3q_0q_1q_2} \quad \begin{pmatrix}
  q_0^2 + q_1q_2 & -(q_2^2 + q_0q_1) & -(q_0q_2 - q_1^2) \\
  -(b^2 + 2ac) & q_0^2 + q_1q_2 & -(q_2^2 + q_0q_1) \\
  -(q_0q_2 - q_1^2) & -(b^2 + 2ac) & q_0^2 + q_1q_2
\end{pmatrix} \\
  f(p)f(q) &= \frac{(p_0^2 + p_1^2 + p_2^2)(q_0^2 + q_1^2 + q_2^2)}{(p_0^2 - p_1^2 + p_2^2 + 3p_0p_1p_2)(q_0^2 - q_1^2 + q_2^2 + 3q_0q_1q_2)} \quad \begin{pmatrix}
  q_0^2 + q_1q_2 & -(q_2^2 + q_0q_1) & -(q_0q_2 - q_1^2) \\
  -(b^2 + 2ac) & q_0^2 + q_1q_2 & -(q_2^2 + q_0q_1) \\
  -(q_0q_2 - q_1^2) & -(b^2 + 2ac) & q_0^2 + q_1q_2
\end{pmatrix}
\end{align*}
\]

Further, we have

\[
\begin{pmatrix}
  (p_1p_2 + p_0^2) & -(p_2^2 + p_1p_0) & -(p_0p_2 - p_1^2) \\
  (p_0p_2 - p_1^2) & (p_1p_2 + p_0^2) & -(p_2^2 + p_0p_1) \\
  (p_2^2 + p_0p_1) & (p_0p_2 - p_1^2) & (p_1p_2 + p_0^2)
\end{pmatrix} \quad \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix} 
\]

where

\[
\begin{align*}
  a_{11} &= (p_1p_2 + p_0^2)(q_1q_2 + q_0^2) - (p_2^2 + p_1p_0)(q_0q_2 - q_1^2) - (p_0p_2 - p_1^2)(q_2^2 + q_1q_0) = A, \\
  a_{12} &= -(p_1p_2 + p_0^2)(q_2^2 + q_1q_0) - (p_2^2 + p_1p_0)(q_1q_2 + q_0^2) - (p_0p_2 - p_1^2)(q_0q_2 - q_1^2) = B, \\
  a_{13} &= -(p_1p_2 + p_0^2)(q_0q_2 - q_1^2) + (p_2^2 + p_1p_0)(q_0q_2 - q_1^2) - (p_0p_2 - p_1^2)(q_1q_2 + q_0^2) = C, \\
  a_{21} &= (p_0p_2 - p_1^2)(q_0q_2 - q_1^2) + (p_2^2 + p_1p_0)(q_0q_2 - q_1^2) - (p_0p_2 - p_1^2)(q_1q_2 + q_0^2) = -C, \\
  a_{22} &= -(p_0p_2 - p_1^2)(q_2^2 + q_1q_0) + (p_1p_2 + p_0^2)(q_1q_2 + q_0^2) - (p_2^2 + p_1p_0)(q_0q_2 - q_1^2) = A, \\
  a_{23} &= -(p_0p_2 - p_1^2)(q_0q_2 - q_1^2) - (p_1p_2 + p_0^2)(q_2^2 + q_1q_0) - (p_2^2 + p_1p_0)(q_1q_2 + q_0^2) = B, \\
  a_{31} &= (p_0p_2 - p_1^2)(q_2^2 + q_1q_0) + (p_1p_2 + p_0^2)(q_0q_2 - q_1^2) + (p_2^2 + p_1p_0)(q_2^2 + q_1q_0) = -B, \\
  a_{32} &= -(p_2^2 + p_1p_0)(q_0q_2 - q_1^2) + (p_0p_2 - p_1^2)(q_1q_2 + q_0^2) + (p_1p_2 + p_0^2)(q_0q_2 - q_1^2) = -C, \\
  a_{33} &= -(p_0p_2 - p_1^2)(q_2^2 + q_1q_0) + (p_1p_2 + p_0^2)(q_1q_2 + q_0^2) - (p_2^2 + p_1p_0)(q_0q_2 - q_1^2) = A.
\end{align*}
\]

Hence,

\[
f(p)f(q) = \frac{(p_0^2 + p_1^2 + p_2^2)(q_0^2 + q_1^2 + q_2^2)}{(p_0^2 - p_1^2 + p_2^2 + 3p_0p_1p_2)(q_0^2 - q_1^2 + q_2^2 + 3q_0q_1q_2)} \quad \begin{pmatrix}
  A_{11} & A_{12} & A_{13} \\
  A_{21} & A_{22} & A_{23} \\
  A_{31} & A_{32} & A_{33}
\end{pmatrix} = \begin{pmatrix}
  A & B & C \\
  -C & A & B \\
  -B & -C & A
\end{pmatrix}.
\]
References


