I. INTRODUCTION

One of the most important, and investigated, problems in physics is the harmonic oscillator. This fact is due to the relevance of this problem in the classical and in the quantum contexts. It is one of the few problems we know how to solve exactly. The solution to this problem is given in many textbooks of classical mechanics, both at introductory and advanced levels, and is given with two arbitrary constants determined according to the initial conditions. The treatment of the damped harmonic oscillator with the presence of external forces can also be found in the literature and its formal treatment. For time dependent external force, requires the formalism of Green’s functions. Generally, undergraduate students at introductory levels do not acquire Green’s functions formalism well.

In this case it is interesting to call attention to the treatmentis harmonic oscillator as in Ref. [1] where the problem is solved without resorting to the standard theory of second order differential equation, a similar treatment as in [2] for the single harmonic oscillator. There are also elaborated treatments for this problem, particularly interesting is the Feynman diagrams technique as in [3] and the Poisson bracket formalism presented as in [4].

In this work, that is intended for undergraduate students and lecturers, we give a general solution to the problem of the damped harmonic oscillator under the influence time-dependent external force. The main advantages of the procedure proposed in this paper are: the constants are fixed from the beginning, there are two initial conditions: starting position and starting velocity in the problem, the mathematical methods employed are accessible for graduate students, we can consider an external force with any time dependence and the method resembles the factorization employed for solving the quantum harmonic oscillator.

II. THEORY

Harmonic oscillator differential equation time dependent external force is:

\[ \ddot{x} + 2b \dot{x} + kx = F(t) \quad (1) \]

with \( b \) is damped coefficient, \( k \) is restoring force, \( m \) is oscillator mass and \( F(t) \) is external force. Harmonic oscillator is calculated by solving eq. 1 with two initial conditions.

We define two constants: \( \alpha \) and \( \beta \), and differential operator \( \partial_t \). Substituting two constants and differential operator, eq. 1 becomes:

\[ \ddot{x} + 2\alpha \dot{x} + \beta x = \partial_t F(t) \quad (2) \]

Or

\[ \ddot{x} + 2\alpha \dot{x} + \beta x = \partial_t F(t) \quad (3) \]

with \( \alpha \) and \( \beta \).

III. ANALYTICAL INVESTIGATION

Mathematical property is:

\[ (a + a\dot{t}) \partial_t f(t) = a f(t) + a \partial_t f(t) \quad (4) \]

that is valid for and complex constants \( a \) and function \( f(t) \) differentiable to \( t \). In harmonic oscillator problems, we can write eq. 3:

\[ \ddot{x} + 2\alpha \dot{x} + \beta x = \partial_t F(t) \quad (5) \]

Now we use mathematical property by substituting, so eq. 5 becomes:

\[ (a + a\dot{t}) \partial_t f(t) = a f(t) + a \partial_t f(t) \quad (6) \]

Integrating to time and input initial conditions, eq. 6 should be written:

\[ (a + a\dot{t}) \partial_t f(t) = a f(t) + a \partial_t f(t) \quad (7) \]

where there are some simple manipulations on the second term on the left hand side of eq. 7. From eq. 7 we have eq. 8:

\[ \dot{x}(t) + \alpha \dot{x}(t) + \frac{\beta x(t)}{\partial_t} = \partial_t F(t) \quad (8) \]

We apply integration to eq. 8 and we have:

\[ \dot{x}(t) + \alpha \dot{x}(t) + \frac{\beta x(t)}{\partial_t} = \partial_t F(t) \quad (9) \]
In eq. 9 time integration is a double integration. We want time integration is not double integration but single integration so we manipulate right side eq. 9.

Partial integration formula is:

\begin{equation}
(10)
\end{equation}

with

We can write:

\begin{equation}
(11)
\end{equation}

Substituting eq. 11 to eq. 9, finally we get:

\begin{equation}
(12)
\end{equation}

The result in eq. 12 is complete agreement with encountered textbooks like as in [5].

References

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