THE BROWNIAN PASSAGE TIME MODEL FOR EARTHQUAKE RECURRENCE PROBABILITIES

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Abstract

Estimation of the time interval until the next large earthquake in a seismic source region is a difficult problem. Conditional probabilities for

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recurrence times of large earthquakes are a reasonable and valid form for estimating the likelihood of future large earthquakes. In this paper, we estimate the interval time for the occurrence of the next large seismic event assuming that the conditional probability of earthquake occurrence is a maximum, provided that a large earthquake has not occurred in the elapsed time since the last large earthquake. We employ a point process probability distribution that is based on a simple physical model of the earthquake recycle, that is the Brownian passage time (BPT) model. BPT model is a renewal model that describes the statistical distribution of rupture time. Application of this model in earthquake forecasting will be given at the end of the paper.

1. Introduction

A probabilistic approach for forecasting the time of the next earthquake on a specific fault segment has been proposed by seismologist (Yilmaz et al. [9] and Ferraes [2, 3]). This study was based on a model of earthquake occurrence assumed that the probability of an earthquake was initially low following a segment-rupturing earthquake, and increased gradually as tectonic processes reloaded the fault. They considered earthquake as a renewal process where large shocks occur repeatedly at approximately regular intervals. Renewal models are frequently used to forecast the next large earthquake on specific fault segments. In this approach, it is assumed that the times between consecutive large earthquakes (recurrence intervals) follow a certain statistical distribution. To forecast the likelihood that a particular earthquake will occur at some time in the future, we required the specification of a probability distribution of failure times. A number of candidate statistical models have been proposed for the computation of conditional probabilistic of future earthquakes, including the Pareto, Rayleigh, and Weibull (Ferraes [2]), gamma and log normal (Ferraes [3]). All of these distributions have been widely discussed as failure model for a broad range of reliability and time to failure problems (Gonzales [4]), although none of them has any particular claim as a proper model for earthquake recurrence. Furthermore, because the predictions obtained from these specific models differ significantly from one another, particularly at times removed from the mean failure time, it is important to consider alternatives to these familiar probability models. In this
paper, we adopted a probability model which is proposed by Matthews as a statistical model for earthquake forecasting that is BPT model. This model has many desirable statistical properties that make it a suitable candidate for describing the statistics of earthquake recurrence.

1.1. Earthquake recurrence probabilities

In this study, we adopted a renewal process for earthquakes. Consider \( R_{t_0} \) as time of the last rupture of the fault segment, \( t_0 \) is the time elapsed since the last rupture to present, and \( \tau \) is the next expected prediction recurrence time. Following Ferraes [3], the time to the next earthquake is the time interval \( \tau \) from the last rupture \( R_{t_0} \) to the next rupture. This interval is a random variable \( T \geq 0 \) with distribution \( F(t_0) = P(T \leq t_0) \). The difference

\[
R(t_0) = 1 - F(t_0) = P(T > t_0)
\]

is the earthquake system reliability and \( F(t_0) \) is the probability that the earthquake system rupture prior to time \( t_0 \), and \( R(t_0) \) is the probability that the earthquake system function at time \( t_0 \). The conditional distribution

\[
F(\tau | T > t_0) = \frac{P(t_0 \leq T \leq \tau)}{P(T \geq t_0)}
\]

is the probability that the earthquake system will fail prior to time \( \tau \). Clearly, \( F(\tau | T > \tau) = 0 \) if \( T < t_0 \) and

\[
F(\tau | T > t_0) = \frac{F(\tau) - F(t_0)}{1 - F(t_0)}, \quad \tau > t_0.
\]

Differentiating with respect to \( \tau \), we obtain the conditional density

\[
\frac{\partial}{\partial \tau} (F(\tau | T \geq t_0)) = f(\tau | T \geq t_0) = \frac{f(\tau)}{1 - F(t_0)}, \quad \tau > t_0.
\]

where \( \int f(\tau | T \geq t_0) d\tau \) is the probability that the earthquake system will rupture in the time interval \((\tau, \tau + d\tau)\), assuming that it has no rupture at time \( t_0 \).
1.2. Brownian passage time model

To forecast the likelihood that a particular earthquake will occur at some time in the future under restrictions requires the specification of a probability distribution of failure times. In this paper, we employ a point process probability distribution the Brownian Passage Time (BPT), which is based on a new time-dependent model. In the BPT model, the failure condition of the fault is described by a state variable that rises from a ground state to the failure state during the earthquake cycle. It proposed as a physically-based renewal model for the earthquake cycle (Matthews et al. [6]). A point process model is used to represent the physics of the earthquake and loading cycle. The probability density for the BPT model is given by: (Ellsworth [1])

$$BPT(\tau) = \sqrt{\frac{\mu}{2\pi \alpha^2 \tau^3}} \exp \left[ -\frac{(\tau - \mu)^2}{2\alpha^2 \mu \tau} \right].$$

The Brownian Passage Time model is a renewal model that describes the statistical distribution of rupture time. In the statistics literature, BPT distribution is known as the Inverse Gaussian Distribution (IGD). As Matthews et al. [6] indicated, the IGD has been known since 1915. However, the distribution only acquired its name in 1945. The IGD distribution is defined by two parameters, the mean time or period between events, $\mu$, and the aperiodicity of the mean time, $\alpha$. The aperiodicity, $\alpha$, is equivalent to the familiar coefficient of variation, defines as standard deviation of the intervals divided by mean rate. It is a measure of the irregularity of the length of the intervals between successive events. Several important properties of the model include: (1) The probability of failure is 0 at $\tau = 0$. (2) As $\tau \to \infty$ the hazard function, or instantaneous failure rate of survivors is finite.

2. Main Result

Assume that $f(\tau)$ is a probability distribution for the recurrence time $\tau$, it is possible to obtain a theoretical conditional probability density model of earthquake occurrence based on a mathematical theory. A reasonable prediction criterion for the occurrence interval $\tau$ between the
last and the next earthquakes is the one which maximizes the conditional probability density \( f(\tau | T \geq t_0) \), i.e.,

\[
\frac{\partial}{\partial \tau} f(\tau | T \geq t_0) = 0. 
\] (6)

So, the estimator of \( \hat{\tau} \) is the solution of equation (6).

Now, if \( \tau \) has a BPT distribution given by equation

\[
f(\tau; \alpha, \mu) = \frac{\mu}{\sqrt{2\pi\alpha^2\tau^3}} \exp\left[ -\frac{(\tau - \mu)^2}{2\alpha^2\mu^2\tau} \right],
\] (7)

then the mean \( E(T) \) and variance \( \text{Var}(T) \) of a BPT distributed variable are

\[
E(T) = \mu \quad \text{and} \quad \text{Var}(T) = \alpha^2\mu^2.
\] (8)

The cumulative distribution for the BPT random variable is

\[
F(\tau) = \int_0^\tau \frac{\mu}{\sqrt{2\pi\alpha^2\tau^3}} \exp\left[ -\frac{(\tau - \mu)^2}{2\alpha^2\mu^2\tau} \right] \, d\tau,
\] (9)

so that,

\[
f(\tau | T \geq t_0) = \frac{\frac{\mu}{\sqrt{2\pi\alpha^2\tau^3}} \exp\left[ -\frac{(\tau - \mu)^2}{2\alpha^2\mu^2\tau} \right]}{1 - \int_0^\tau \frac{\mu}{\sqrt{2\pi\alpha^2\tau^3}} \exp\left[ -\frac{(\tau - \mu)^2}{2\alpha^2\mu^2\tau} \right]}.
\] (10)

because the observed elapsed time \( t_0 \) is a constant, the cumulative distribution of BPT random variable in equation (9) is a constant equal to \( C \). Thus, we can write the BPT conditional probability density of earthquake occurrence as follows:

\[
f(\tau | T \geq t_0) = W \frac{\mu}{\sqrt{2\pi\alpha^2\tau^3}} \exp\left[ -\frac{(\tau - \mu)^2}{2\alpha^2\mu^2\tau} \right],
\] (11)

where \( W = \frac{1}{1 - C} \). Now, we proceed to find the recurrence time \( \hat{\tau} \) which maximizes the BPT conditional probability density of earthquake
occurrence in (11). We find the maximum of \( f(\tau | T \geq t_0) \) by calculating its derivative and setting it equals to zero value

\[
\frac{\partial f(\tau)}{\partial \tau} = \frac{\sqrt{2} \exp\left(-\frac{(\tau - \mu)^2}{2\sigma^2}\right) \left(-3\alpha^2 \sigma^2 - \tau^2 + \mu^2\right)}{4\alpha^4 \tau^5 \sqrt{\pi \alpha^2 \tau^3}}.
\]

(12)

Thus, we have the critical points as follows:

\[
\hat{\tau}_1 = \frac{3}{2} \alpha^2 \mu + \frac{1}{2} \mu (\sqrt{9\alpha^4 + 4})
\]

(13)

and

\[
\hat{\tau}_2 = \frac{3}{2} \alpha^2 \mu - \frac{1}{2} \mu (\sqrt{9\alpha^4 + 4})
\]

(14)

because of \( \tau \) is positif, so the solution of equation (12) is \( \hat{\tau}_1 \).

Furthermore, criterion of maximum conditional probability will be verified. Consider equation (12) for interval \([0, \infty)\). Since \( f'(\tau) > 0 \) for all \( \tau \) in \([0, \hat{\tau}_1]\), \( f \) is increasing on \((0, \hat{\tau}_1]\) by the monotonicity theorem. Again, since \( f'(x) < 0 \) for all \( x \) in \((\hat{\tau}_1, \infty)\), \( f \) is decreasing on \([\hat{\tau}_1, \infty)\). Thus, we conclude that \( f(\hat{\tau}_1) \) is a maximum.

Using equation (13) we can forecast the BPT recurrence of the next expected large earthquake event. It should be noted that in the theory of conditional probability density by definition of recurrence time the predicted recurrence time includes the elapsed time \( t_0 \). However, equation (13) indicates that for the BPT model predicted recurrence time \( \hat{\tau} \) is independent of the elapsed time \( t_0 \) since the last large earthquake.

2.1. Case study

As an illustration, we are possible to evaluate and simulate the temporal probability of earthquake occurrence using seismic data. Using the data set taken from Engdahl catalogue (1964-2000) with magnitude \( M > 6 \) in a range of longitude 119 to 126 and latitude –9 to –8, we know that the last rupture of the fault happened at 1995.37. The value of mean is 4.98 and the value of variance is 20.125.
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We calculate $\mu$ and $\alpha$ by solving equation (8) simultaneously, we find $\mu = 4.98$ and $\alpha = 0.901$. To predict the recurrence time $\hat{\tau}$ using BPT model, we substitute the values of the parameters to equation (13) and obtain $\hat{\tau}_{BPT} = 0.082$ years. Finally, we predict the occurrence time of the next large earthquake ($M > 6$). To do this, we add the BPT recurrence time $\hat{\tau} = 0.082$ years and the occurrence time of the last observed earthquake 1995.37. The result of simulation shows that the next large earthquake may occur approximately before 1995, $37 + 0.082 = 1995.452$ or before June 1995.

2.2. Prediction error for the BPT prediction

Following Klugman [5], any predictor of $\tau$ has a square error:

$$\varepsilon^2 = E[(\tau - \hat{\tau})^2] = \text{Var}[\tau] + (\mu_\tau - \hat{\tau})^2,$$

where $\mu_\tau$ is the mean of the sample $\tau_i$.

Using the data set, we have calculated $\mu_\tau = 4.98$ (years) and $\text{Var}[\tau] = 20.125$ (years)$^2$. Thus, we estimate the square error ($\varepsilon^2$) for the predicted BPT recurrence time $\hat{\tau}_{BPT} = 0.082$ years as follows:

$$\varepsilon^2 = 20.125 + (4.98 - 0.082)^2 = 44.115$$

or equivalently, $\varepsilon = 6.642$ years. Using this value of the prediction error, the BPT occurrence time of the next expected large earthquake ($M > 6$) in the Nusatenggara region can be written by $t_{BPT} = 1995.452 \pm 6.642$ years.

3. Discussion

We have adopted the BPT model for earthquake recurrence because Ellsworth [1] carried out that the properties of the BPT model not only satisfy the spirit of the hypothesis proposed by Utsu [8] and Rikitake [7], but also provide more realistic asymptotic behavior of the failure rate than the alternative models. In this paper, we have estimated earthquake recurrence time for BPT model and applied to forecasting of earthquake
occurrence in Nusa Tenggara region Indonesia. These forecasts were based on earthquake data only with no explicit use of tectonic, geologic, or geodetic information. Using the historical earthquake occurrence data from Engdahl catalogue (1935-2000) with $M > 6$, we simulated and found that if the last rupture of the fault happened at 1995,37 (before May 1995), then the next large earthquake will occur at 1995, 452 ± 6, 642 years.

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