Convolution and Correlation Theorems for Ambiguity Functions

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\begin{abstract}
We introduce the definition of convolution and correlation of the Fourier transform, then obtain general convolution and correlation theorems of the ambiguity functions (AF).

\textbf{Keywords:} Fourier transform, ambiguity function, convolution theorem, correlation theorem
\end{abstract}

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\section{1 Introduction}

The ambiguity function (AF) is first introduced by Woodward in 1953. The AF plays as a major role for the mathematical analysis of sonar and the radar signals. A number of useful properties of the AF are already known, including nonlinearity, shift, modulation, differentiation, and the energy density spectrum [1, 2, 4, 5, 6].

On the other hand the convolution and correlation theorems in the AF domain are fundamental topics in theory and application of the AF. In [3, 7], the authors derived convolution and correlation theorems for the Wigner-Ville distribution, but they have not discussed the convolution and correlation theorems for the AF. In this paper we establish general convolution and correlation of the ambiguity functions (AF). We find that they are significantly different from the convolution and correlation theorems for the Wigner-Ville distribution.

The paper is organized as follows: The preliminaries about the Fourier (FT) and its important properties are mainly presented in §2. We also discuss convolution and correlation of the FT, which will be used in the next section. The construction of the AF and its important properties such as the modulation, translation, and Parseval’s formulas are presented in §3. The theorem of the convolution in the AF domain is proposed in §4. We finally establish theorem of two complex signals in the AF domain in §5.

\section{2 Preliminaries}

Let us introduce the convolution and correlation definitions and relationship among convolution, correlation and the Fourier transform (FT). For a complex-valued function $f$ defined on $\mathbb{R}$, the complex conjugate $\bar{f}$ of $f$ is given by $\bar{f}(x) = f(x)\overline{.}$ Denote by $L^p(\mathbb{R})$, $1 < p < +\infty$, the Banach space of functions for which the $p$-th power of the absolute value is Lebesgue integrable.

\begin{definition}[Fourier Transform] Let $f \in L^1(\mathbb{R})$. The Fourier transform of $f$ is defined by
\begin{equation}
\mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}} f(x)e^{-i\omega x}\,dx.
\end{equation}
\end{definition}

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By the inner product notation of \(L^2(\mathbb{R})\), the FT can be expressed as \(\hat{f}(\omega) = \langle f, e^{i\omega x} \rangle\). The norm naturally induced by the inner product is given by \(\|f\|_2 = \sqrt{\langle f, f \rangle}\).

**Theorem 2.1 (Inversion Formula).** For \(g \in L^1(\mathbb{R})\), the inverse FT of \(g\) is given by

\[
\mathcal{F}^{-1}[g](x) = \frac{1}{2\pi} \int_\mathbb{R} g(\omega) e^{i\omega x} d\omega.
\]

Note that Dirac’s delta can be represented by

\[
\delta(x) = \frac{1}{2\pi} \int_\mathbb{R} e^{i\omega x} d\omega.
\]

**Definition 2.2 (Convolution).** For \(f, g \in L^1(\mathbb{R})\), the convolution of \(f\) and \(g\), denoted \(f * g\), is defined by

\[
(f * g)(x) = \int_\mathbb{R} f(t) g(x-t) dt.
\]

**Theorem 2.2.** Suppose that \(f, g \in L^1(\mathbb{R})\). Then we have

\[
\mathcal{F}\{f * g\}(\omega) = \mathcal{F}\{f\}(\omega)\mathcal{F}\{g\}(\omega),
\]

\[
(f * g)(x) = \frac{1}{2\pi} \int_\mathbb{R} \mathcal{F}\{f\}(\omega)\mathcal{F}\{g\}(\omega) e^{i\omega x} d\omega.
\]

Next, let us examine how the FT behaves under correlations. First, we give the definition of the correlation of two complex functions.

**Definition 2.3 (Correlation).** Suppose that \(f, g \in L^1(\mathbb{R})\), the correlation of \(f\) and \(g\) is defined by

\[
(f * g)(x) = \int_\mathbb{R} \overline{f}(t)g(t+x) dt.
\]

**Theorem 2.3.** Suppose that \(f, g \in L^1(\mathbb{R})\). Then the FT of correlation of \(f\) and \(g\) is given by

\[
\mathcal{F}\{f * g\}(\omega) = \mathcal{F}\{\overline{f}\}(-\omega)\mathcal{F}\{g\}(\omega),
\]

or, equivalently,

\[
(f * g)(x) = \frac{1}{2\pi} \int_\mathbb{R} \mathcal{F}\{\overline{f}\}(-\omega)\mathcal{F}\{g\}(\omega) e^{i\omega x} d\omega.
\]

Define the translation operator \(\tau_k\) and the modulation operator \(M_\omega\) by \(\tau_k f(t) = f(t-k)\) and \(M_\omega f(t) = e^{i\omega t} f(t)\), respectively. These operators are unitary on \(L^2(\mathbb{R})\).

### 3 Ambiguity Function (AF)

We will give the definition of the cross AF, auto AF and relationship among them and the Fourier transform (FT).

**Definition 3.1.** For \(f, g \in L^2(\mathbb{R})\), the cross ambiguity function of \(f\) and \(g\) is given by

\[
A_{f,g}(t,\omega) = \int_\mathbb{R} f\left(x + \frac{t}{2}\right) \overline{g}\left(x - \frac{t}{2}\right) e^{-i\omega x} dx,
\]

provided the integral exists.
Denote by $V_g f(t, \omega)$, the windowed Fourier transform of $f$ with respect to the window function $g$. By the change of variables $x + \frac{t}{2} = \tau$, equation (10) can be written in the form

$$A_{f,g}(t, \omega) = \int_{\mathbb{R}} f(t) g(\tau - t) e^{-i2\omega \tau} d\tau$$

$$= e^{\frac{it}{2}} \int_{\mathbb{R}} f(\tau) g(\tau - t) e^{-i2\omega \tau} d\tau$$

$$= e^{\frac{it}{2}} V_g f(t, \omega).$$

Using (1), we can get

$$A_{f,g}(t, \omega) = \mathcal{F}\{R_{f,g}(t, x)})(\omega),$$

where

$$R_{f,g}(t, x) = f\left(t + \frac{x}{2}\right) g\left(-\frac{x}{2}\right).$$

Applying Dirac's delta representation (3), we get the FT of the AF with respect to $\omega$ (see [2, 3]) as

$$\tilde{A}_{f,g}(t, \sigma) = 2\pi f\left(-\sigma + \frac{t}{2}\right) \tilde{g}\left(-\sigma - \frac{t}{2}\right).$$

The following lemma gives a relation between $A_{f,g}$ and $A_{f,f}$.

**Lemma 3.1.** For $f, g \in L^2(\mathbb{R})$, we have

$$A_{f,g}(t, \omega) = \frac{1}{2\pi} A_{f,f}(\omega, -t).$$

**Proof.** By Parseval’s formula, the ambiguity function can be represented as

$$A_{f,g}(t, \omega) = \int_{\mathbb{R}} f\left(x + \frac{t}{2}\right) e^{-i(\omega/2)x} \overline{g\left(x - \frac{t}{2}\right)} e^{i(\omega/2)x} dx$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}\left[f\left(x + \frac{t}{2}\right) e^{-i(\omega/2)x}\right] \mathcal{F}\left[g\left(x - \frac{t}{2}\right) e^{i(\omega/2)x}\right] d\xi.$$

Since

$$\mathcal{F}\left[f\left(x + \frac{t}{2}\right) e^{-i(\omega/2)x}\right] = e^{i(t/2)(\xi + (\omega/2))} \hat{f}(\xi + \frac{\omega}{2}),$$

$$\mathcal{F}\left[g\left(x - \frac{t}{2}\right) e^{i(\omega/2)x}\right] = e^{-i(t/2)(\xi - (\omega/2))} \tilde{g}(\xi - \frac{\omega}{2}),$$

it implies that

$$A_{f,g}(t, \omega) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi + \frac{\omega}{2}) \tilde{g}(\xi - \frac{\omega}{2}) e^{i\xi} d\xi$$

$$= \frac{1}{2\pi} A_{f,f}(\omega, -t).$$

When $f = g$, we abbreviate $A_{f,g}(t, \omega)$ to $A_f(t, \omega)$, that is,

$$A_f(t, \omega) = \int_{\mathbb{R}} f\left(x + \frac{t}{2}\right) \hat{f}\left(x - \frac{t}{2}\right) e^{-i\omega x} dx,$$

which is called *auto ambiguity function*. Usually both the cross AF and the auto AF are simply referred as the AF. We should remember that the AF is a nonlinear time frequency transform, because the integrand contains the multiplication of $f$ and $\hat{f}$. 

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3.1 Important Properties of Ambiguity Function

Some important properties of the ambiguity function (AF) are summarized in the following theorems, which will be important later.

1. COMPLEX CONJUGATION

**Theorem 3.1.** For arbitrary \( f, g \in L^2(\mathbb{R}) \), we have

\[
\hat{A}_{f,g}(t,\omega) = A_{g,f}(-t,-\omega).
\]

**Proof.** It can easily be derived using basic properties of complex numbers. \( \square \)

2. INVERSION FORMULA

**Theorem 3.2.** Let \( f, g \in L^2(\mathbb{R}) \). Then every complex function \( f \) can be fully reconstructed, we have

\[
f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\frac{i}{2} \omega t} \hat{A}_{f,g}(t,\omega) \, d\omega.
\]

**Proof.** By (12), we have

\[
\int_{\mathbb{R}} e^{-i\omega t} \hat{A}_{f,g}(t,\omega) \, d\omega = 2\pi f \left( -\sigma + \frac{t}{2} \right) \tilde{g} \left( -\sigma - \frac{t}{2} \right).
\]

Putting \(-\sigma + \frac{t}{2} = t_1 \) and \(-\left(\sigma + \frac{t}{2}\right) = t_2 \), we obtain

\[
\int_{\mathbb{R}} e^{-i\omega \left( t_1 + t_2 \right)} \hat{A}_{f,g}(t_1 - t_2,\omega) \, d\omega = 2\pi f(t_1) \tilde{g}(t_2).
\]

Choosing \( t_1 = t \) and \( t_2 = 0 \), we have

\[
\int_{\mathbb{R}} e^{i\omega t} \hat{A}_{f,g}(t,\omega) \, d\omega = 2\pi f(t) \hat{g}(0).
\]

Therefore,

\[
f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\frac{i}{2} \omega t} \hat{A}_{f,g}(t,\omega) \, d\omega.
\]

which completes the proof. \( \square \)

3. TIME MARGINAL

**Theorem 3.3.** For \( f \in L^2(\mathbb{R}) \), we have

\[
\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega t} \hat{A}_{f}(0,\omega) \, d\omega = |f(t)|^2.
\]

**Proof.** Since

\[
A_f(0,\omega) = \int_{\mathbb{R}} |f(x)|^2 e^{-i\omega x} \, dx = \mathcal{F}[|f(x)|^2],
\]

we have

\[
|f(x)|^2 = \mathcal{F}^{-1}[A_f(0,\omega)] = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega t} \hat{A}_{f}(0,\omega) \, d\omega.
\]

\( \square \)
4. Frequency Marginal

**Theorem 3.4.** For $f \in L^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} e^{-i\omega t} A_f(t,0) \, dt = |\hat{f}(\omega)|^2.$$  \hfill (19)

**Proof.** For $\hat{f} = \hat{g}$ and $s = -t$, Lemma 3.1 implies

$$A_f(\omega, s) = 2\pi A_f(-s, \omega).$$

Applying Theorem 3.4 to $\hat{f}$ and we have

$$|\hat{f}(\xi)|^2 = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi s} A_f(0,s) \, ds$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi s} 2\pi A_f(-s,0) \, ds$$

$$= \int_{\mathbb{R}} e^{-i\xi t} A_f(t,0) \, dt.$$

\[\square\]

5. Translation

**Theorem 3.5.** Let $f, g \in L^2(\mathbb{R})$. Then we have

$$A_{\tau_s f, \tau_s g}(t,\omega) = e^{-i\omega k} A_{f,g}(t,\omega).$$ \hfill (20)

In particular,

$$A_{\tau_s f}(t,\omega) = e^{-i\omega k} A_f(t,\omega).$$ \hfill (21)

6. Modulation

**Theorem 3.6.** Let $f, g \in L^2(\mathbb{R})$. Then we have

$$A_{M_{\omega t} f, M_{\omega t} g}(t,\omega) = e^{i\omega t} A_{f,g}(t,\omega).$$ \hfill (22)

Moreover, we also have

$$A_{M_{\omega t} f}(t,\omega) = e^{i\omega t} A_f(t,\omega).$$ \hfill (23)

7. Translation and Modulation

**Theorem 3.7.** Let $f, g \in L^2(\mathbb{R})$. Then

$$A_{M_{\omega t} \tau_s f, M_{\omega t} \tau_s g}(t,\omega) = e^{i(\omega t - \omega k)} A_{f,g}(t,\omega).$$ \hfill (24)

**Proof.** Applying Theorem 3.6 and Theorem 3.5, we have

$$A_{M_{\omega t} \tau_s f, M_{\omega t} \tau_s g}(t,\omega) = e^{i\omega t} A_{\tau_s f, \tau_s g}(t,\omega)$$

$$= e^{i\omega t} e^{-i\omega k} A_{f,g}(t,\omega)$$

$$= e^{i(\omega t - \omega k)} A_{f,g}(t,\omega).$$

\[\square\]

8. Parseval's Formulas
Theorem 3.8. For \( f_1, g_1, f_2, g_2 \in L^2(\mathbb{R}) \), we have

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} A_{f_1, g_1}(t, \omega)A_{f_2, g_2}(t, \omega) \, dt \, d\omega = 2\pi (f_1, f_2)(g_1, g_2).
\]  (25)

In particular, we have

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} |A_{f, g}(t, \omega)|^2 \, dt \, d\omega = 2\pi \|f\|_2 \|g\|_2.
\]  (26)

and

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} A_{f}(t, \omega)A_{g}(t, \omega) \, dt \, d\omega = 2\pi (f, g)^2.
\]  (27)

Proof. Applying Parseval’s formula to \( \omega \)-integral, we have

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} A_{f_1, g_1}(t, \omega)A_{f_2, g_2}(t, \omega) \, dt \, d\omega
\]

\[
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} F\{R_{f_1, g_1}(t, x)\}(\omega)\overline{F\{R_{f_2, g_2}(t, x)\}(\omega)} \, d\omega \right) \, dt
\]

\[
= \int_{\mathbb{R}} \left( 2\pi \int_{\mathbb{R}} R_{f_1, g_1}(t, x)\overline{R_{f_2, g_2}(t, x)} \, dx \right) \, dt
\]

\[
= 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} f_1 \left( x + \frac{t}{2} \right) g_1 \left( x - \frac{t}{2} \right) g_2 \left( x + \frac{t}{2} \right) \overline{f_2} \left( x - \frac{t}{2} \right) \, dx \, dt.
\]

By the change of variables: \( y = x + \frac{t}{2} \) and \( z = x - \frac{t}{2} \), we have

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} A_{f_1, g_1}(t, \omega)A_{f_2, g_2}(t, \omega) \, dt \, d\omega
\]

\[
= 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} f_1(y) g_1(z) g_2(z) \overline{f_2}(y) \, dy \, dz
\]

\[
= 2\pi (f_1, f_2)(g_1, g_2).\]

This proves the proof of (25).

\[\square\]

4 Convolution Theorem for Ambiguity Function

Convolution plays an important role in signal processing and image processing, such as system identification, edge detection, sharpening and smoothing of images. It is well known that the classical convolution in time domain corresponds to the product in the Fourier domain. In this section, we shall consider the convolution theorem in the AF domain. It can be considered as an extension of the convolution from the FT to the AF domain.

Theorem 4.1. For \( f, g \in L^2(\mathbb{R}) \),

\[
A_{f \ast g}(t, \omega) = \int_{\mathbb{R}} A_f(u, \omega)A_g(t - u, \omega) \, du.
\]  (28)

Proof. It follows from (10) that

\[
A_{f \ast g}(t, \omega) = \int_{\mathbb{R}} (f \ast g) \left( x + \frac{t}{2} \right) \overline{(f \ast g) \left( x - \frac{t}{2} \right)} e^{-i\omega x} \, dx.
\]

By (4), we have

\[
A_{f \ast g}(t, \omega) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(\tau)g \left( x + \frac{t}{2} - \tau \right) \, d\tau \right) \left( \int_{\mathbb{R}} \overline{\hat{f}(\xi)}\overline{g} \left( x - \frac{t}{2} - \xi \right) \, d\xi \right) e^{-i\omega x} \, dx
\]
Or, equivalently,
\[
A_{f*g}(t,\omega) = \int_{\mathbb{R}^2} f(\tau)g\left(x + \frac{t}{2} - \tau\right) \tilde{f}(\xi) \tilde{g}\left(x - \frac{t}{2} - \xi\right) e^{-i\omega x} d\tau d\xi dx.
\]
Put \(\tau = p + \frac{u}{2}, \xi = p - \frac{u}{2}, x = p + q\), that is, \(p = \frac{\tau + \xi}{2}, u = \tau - \xi, q = x - \frac{\tau + \xi}{2}\). Then,
\[
j(\tau, \xi, x) = \frac{\partial(u, p, q)}{\partial(\tau, \xi, x)} = \begin{vmatrix}
\frac{\partial u}{\partial \tau} & \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial x} \\
\frac{\partial q}{\partial \tau} & \frac{\partial q}{\partial \xi} & \frac{\partial q}{\partial x} \\
\frac{\partial p}{\partial \tau} & \frac{\partial p}{\partial \xi} & \frac{\partial p}{\partial x}
\end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
-\frac{1}{2} & -\frac{1}{2} & 1 \end{vmatrix} = 1.
\]
We further have
\[
A_{f*g}(t,\omega) = \int_{\mathbb{R}} A_f(u,\omega)A_g(t-u,\omega) du.
\]

**Theorem 4.2** (Convolution with respect to both variables). For \(f, g \in L^2(\mathbb{R})\), we have
\[
(A_f * A_g)(a, b) = -2\pi (M_{\frac{\tau}{2}}g,f)(M_{-\frac{\tau}{2}}g,f).
\]

Proof. It follows from (4) that
\[
(A_f * A_g)(a, b) = \int_{\mathbb{R}^2} A_f(t,\omega)A_g(a-t, b-\omega) d\omega dt.
\]
Using (12), we have
\[
\int_{\mathbb{R}^2} A_f(t,\omega)A_g(a-t, b-\omega) d\omega dt = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{A}_f(t,\sigma)\hat{A}_g(a-t, \sigma) e^{i\sigma b} d\sigma.
\]
Applying (13) to the right-hand side of (29), we have
\[
\int_{\mathbb{R}^2} A_f(t,\omega)A_g(a-t, b-\omega) d\omega dt = \frac{1}{2\pi} \int_{\mathbb{R}} 2\pi f\left(-\sigma + \frac{t}{2}\right) \tilde{f}\left(-\sigma - \frac{t}{2}\right) 2\pi g\left(-\sigma + \frac{a-t}{2}\right) \tilde{g}\left(-\sigma - \frac{a-t}{2}\right) e^{i\sigma b} d\sigma
\]
\]
\[ = 2\pi \int_{\mathbb{R}} f(-\sigma + \frac{t}{2}) \bar{f}(-\sigma - \frac{t}{2}) g(-\sigma + \frac{a-t}{2}) \bar{g}(-\sigma - \frac{a-t}{2}) e^{i\sigma} d\sigma \]

\[ = 2\pi \int_{\mathbb{R}} f(-\sigma + \frac{t}{2}) \bar{f}(-\sigma - \frac{t}{2}) g\left(\frac{a}{2} + \left(-\sigma - \frac{t}{2}\right)\right) \bar{g}\left(\frac{a}{2} + \left(-\sigma + \frac{t}{2}\right)\right) e^{i\sigma} d\sigma. \]

Setting \( \sigma = -\frac{t}{2} - \tau \), we obtain

\[ \int_{\mathbb{R}} \mathcal{A}_f(t,\omega) \mathcal{A}_g(a-t, b-\omega) \, d\omega \, dt \]

\[ = -2\pi \int_{\mathbb{R}} f(t + \tau) \bar{f}(\tau) g\left(\frac{a}{2} + \tau\right) \bar{g}\left(-\frac{a}{2} + t + \tau\right) e^{i\frac{\omega t}{2}} e^{-i\tau} \, d\tau. \quad (30) \]

Integrating both sides of (30) with respect to \( dt \), we obtain

\[ \int_{\mathbb{R}} \mathcal{A}_f(t,\omega) \mathcal{A}_g(a-t, b-\omega) \, d\omega \, dt \]

\[ = -2\pi \int_{\mathbb{R}^2} f(t + \tau) \bar{f}(\tau) g\left(\frac{a}{2} + \tau\right) \bar{g}\left(-\frac{a}{2} + t + \tau\right) e^{-i\frac{\omega t}{2}} e^{-i\tau} \, dt \, d\tau. \quad (31) \]

Substituting \( t + \tau = u \) into (31), we have

\[ \int_{\mathbb{R}^2} \mathcal{A}_f(t,\omega) \mathcal{A}_g(a-t, b-\omega) \, d\omega \, dt \]

\[ = -2\pi \int_{\mathbb{R}^2} f(u) \bar{f}(s) g\left(\frac{a}{2} + s\right) \bar{g}\left(-\frac{a}{2} + u\right) e^{-i\frac{\omega u(s)}{2}} e^{-ibs} \, ds \, du \]

\[ = -2\pi \int_{\mathbb{R}^2} f(u) \bar{f}(s) e^{-ib\frac{s}{2}} \tau\frac{\omega}{2} g(u) e^{-i\frac{b\omega}{2}} e^{-i\tau} \, ds \, du \]

\[ = -2\pi \int_{\mathbb{R}^2} f(u) e^{-ib\frac{s}{2}} \tau\frac{\omega}{2} \bar{g}(u) du \int_{\mathbb{R}} \bar{f}(s) e^{-i\frac{b\omega}{2}} \tau\frac{-\omega}{2} g(s) ds \]

\[ = -2\pi \int_{\mathbb{R}} f(u) M_{\frac{\omega}{2}} g(u) du \int_{\mathbb{R}} \bar{f}(s) M_{-\frac{\omega}{2}} g(s) ds \]

\[ = -2\pi \int_{\mathbb{R}} M_{\frac{\omega}{2}} g(u) f(u) du \int_{\mathbb{R}} M_{-\frac{\omega}{2}} g(s) \bar{f}(s) ds \]

\[ = -2\pi \left( M_{\frac{\omega}{2}} g, f \right) \left( M_{-\frac{\omega}{2}} g, f \right). \]

\[ \Box \]

5 Correlation Theorem for Ambiguity Function

The correlation and convolution are closely related, because the correlation can be regarded as the conjugate of convolution. In this section, we establish the correlation in the AF domain.

**Theorem 5.1.** Let \( f, g \in L^2(\mathbb{R}) \). Then

\[ \mathcal{A}_{f*g}(t,\omega) = \int_{\mathbb{R}} \mathcal{A}_f(u,\omega) \mathcal{A}_g(u + t, \omega) \, du. \quad (32) \]

**Proof.** Using (10), we obtain

\[ \mathcal{A}_{f*g}(t,\omega) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \bar{f}(\tau) g \left( \tau + \left(x + \frac{t}{2}\right) \right) d\tau \right) \int_{\mathbb{R}} f(\xi) g \left( \xi + \left(x - \frac{t}{2}\right) \right) d\xi e^{-i\omega x} \, dx. \]

We put

\[ \tau = p + \frac{u}{2}, \quad \xi = p - \frac{u}{2}, \quad x = q - p, \]
that is,  
\[ p = \frac{\tau + \xi}{2}, \quad u = \tau - \xi, \quad q = x + \frac{\tau + \xi}{2}. \]

Since  
\[ \mathcal{J}(\tau, \xi, x) = \frac{\partial(u, p, q)}{\partial(\tau, \xi, x)} = \begin{bmatrix} \frac{\partial u}{\partial \tau} & \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial x} \\ \frac{\partial p}{\partial \tau} & \frac{\partial p}{\partial \xi} & \frac{\partial p}{\partial x} \\ \frac{\partial q}{\partial \tau} & \frac{\partial q}{\partial \xi} & \frac{\partial q}{\partial x} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} = 1, \]

it implies that  
\[
A_{f,g}(t, \omega) = \int_{\mathbb{R}} A_f(u, \omega) A_g(u + t, \omega) du.
\]

**Theorem 5.2** (Correlation with Respect to Both Variables). Let \( f, g \in L^2(\mathbb{R}) \). Then  
\[
(A_f \ast A_g)(a, b) = 2\pi (\hat{f}, M_{\frac{1}{2}} \hat{g})(f, M_{\frac{1}{2}} \tau_{-\frac{1}{2}} g).
\]

**Proof.** From (10), we have  
\[
(A_f \ast A_g)(a, b) = \int_{\mathbb{R}} A_f(t, \omega) A_g(a - t, b - \omega) d\omega.
\]

Applying (12), we have  
\[
\int_{\mathbb{R}} A_f(t, \omega) A_g(a + t, b + \omega) d\omega = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{A}_f(t, \sigma) \hat{A}_g(a + t, \sigma) e^{i\sigma t} d\sigma.
\]

By (15), we obtain  
\[
\int_{\mathbb{R}} A_f(t, \omega) A_g(a + t, b + \omega) d\omega = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{2\pi f(-\sigma + t/2)}{2\pi g(-\sigma - a + t/2)} \frac{2\pi g(-\sigma + a + t/2)}{2\pi f(-\sigma - t/2)} e^{i\sigma t} d\sigma.
\]

Substitute \( \sigma = \frac{1}{2} - t \) into the above equation, we have  
\[
\int_{\mathbb{R}} A_f(t, \omega) A_g(a + t, b + \omega) d\omega
\]
Integrating both sides of the above equation with respect to $dt$ yields

$$\int_{R^2} A_f(t, \omega) A_g(a + t, b + \omega) \, d\omega \, dt$$

$$= -2\pi \int_{R^2} f(\tau) \bar{f}(\tau - t) g \left( \frac{a}{2} + \tau \right) \bar{g} \left( -\frac{a}{2} + (\tau - t) \right) e^{i\theta(\frac{1}{2} - \tau)} \, d\tau \, dt$$

$$= -2\pi \int_{R^2} f(\tau) \bar{f}(\tau - t) g \left( \frac{a}{2} + \tau \right) \bar{g} \left( -\frac{a}{2} + (\tau - t) \right) e^{i\theta y} e^{-i\theta r} \, d\tau \, dt.$$

Substituting $\tau - t = u$ into the above expression, we have

$$\int_{R^2} A_f(t, \omega) A_g(a + t, b + \omega) \, d\omega \, dt$$

$$= 2\pi \int_{R^2} f(u) \bar{f}(u + a) g \left( \frac{a}{2} + u \right) \bar{g} \left( u - \frac{a}{2} \right) e^{i\theta(u - \frac{a}{2})} e^{-i\theta u} \, du \, dr$$

$$= 2\pi \int_{R} f(u) e^{i\frac{\theta}{2} u} \left( u - \frac{a}{2} \right) \, du \int_{R} f(\tau) e^{i\frac{\theta}{2} \tau} \left( \tau + \frac{a}{2} \right) \, d\tau$$

$$= 2\pi (f, M_{\frac{\theta}{2}}^* g)(f, M_{\frac{\theta}{2}}^* g),$$

which completes the proof. \hfill \Box

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