AXISYMMETRIC LOADING OF A CLASS OF INHOMOGENEOUS TRANSVERSELY ISOTROPIC HALF-SPACES WITH QUADRATIC ELASTIC MODULI

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Summary

Some axisymmetric problems are considered for a class of inhomogeneous anisotropic elastic materials for which the elastic moduli vary continuously as a quadratic function of the spatial coordinates. For a restricted class of transversely isotropic materials, loading of a half-space and a layer on a rigid foundation are considered and integral expressions for the displacement and stress are obtained. Particular attention is given to cases when the values of the elastic moduli near to the plane boundary of the half-space and the layer approach zero. Numerical results are obtained for some example transversely isotropic and isotropic materials.

1. Introduction

The solution of axially symmetric problems for an inhomogeneous half-space in which the elastic moduli vary continuously with the spatial coordinates has attracted the attention of a number of authors. Examples of early work in this area include papers by Gibson (1), Gibson et al. (2) and Awojobi and Gibson (3) who considered some axisymmetric contact problems for a half-space in which the elastic moduli are a linear function of the perpendicular distance from the plane boundary, while Mossakovskii (4) considered an axisymmetric problem with the elastic moduli varying exponentially with the perpendicular distance from the boundary. Other examples of solutions to axisymmetric problems for half-spaces with continuously varying elastic moduli include the work of Puro (5) who considered problems with a power law variation and Selvadurai (6) and Vrettos (7) who considered problems with an exponential variation in the elastic moduli. Also the recent review paper by Selvadurai (8) contains a number of references to papers that have addressed problems in this area in the latter half of the 20th century. Other related papers of interest involving inhomogeneous materials include the work of Martin et al. (9) and Wang et al. (10).

This paper is concerned with the solution of some axisymmetric problems for a restricted class of transversely isotropic inhomogeneous elastic materials. The elastic material occupies a half-space

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or an elastic layer that adheres to a rigid base. The problems considered involve determining the displacement and stress in the elastic material due to a specified normal axisymmetric load on the plane boundary of the half-space and the elastic layer. Previous solutions to problems of this type have generally involved materials in which the elastic moduli are a linear or exponential function of the normal distance from the plane boundary or proportional to a power of this normal distance. The current work is concerned with the case when the elastic moduli are a quadratic function of the spatial variables. Solutions to the problems under consideration are obtained in terms of integrals that yield analytical information and numerical values for the displacement and stress. The case when the inhomogeneous material has very small elastic moduli in a narrow region close to the plane surface is considered. In this case, the results obtained exhibit some similar characteristics to the results obtained by Gibson (1) and Gibson et al. (2) for incompressible isotropic materials in which the elastic parameters are a linear function of the spatial variables.

The analysis is for a restricted class of transversely isotropic materials and through a limiting procedure also yields numerical results for the relevant class of isotropic materials.

2. Statement of the problem
Referred to a Cartesian frame \( O_{x1x2x3} \), consider an anisotropic elastic body. Let the body occupy the region \( \Omega_1 \), which consists either of the half-space \( x_3 > 0 \) or the slab lying in the region \( 0 < x_3 < h \), where \( h \) is a constant. On the plane boundary \( x_3 = 0 \), the axisymmetric stress is specified and the slab adheres to a rigid foundation so that for the slab, the displacement is zero on \( x_3 = h \). The problem is to determine the stress and displacement throughout the elastic material.

3. Basic equations
The equilibrium equations governing small deformations of an inhomogeneous anisotropic elastic material may be written in the form

\[
\frac{\partial}{\partial x_j} \left[ c_{ijkl}(x) \frac{\partial u_k(x)}{\partial x_l} \right] = 0, \tag{3.1}
\]

where \( i, j, k, l = 1, 2, 3 \), \( x = (x_1, x_2, x_3) \), \( u_k \) denotes the displacement, \( c_{ijkl}(x) \) the elastic moduli and the repeated summation convention (summing from 1 to 3) is used for repeated Latin suffices. The stress displacement relations are given by

\[
\sigma_{ij}(x) = c_{ijkl} \frac{\partial u_k}{\partial x_l}, \tag{3.2}
\]

and the stress vector \( P_i \) on a boundary with outward pointing normal \( n = (n_1, n_2, n_3) \) is defined as

\[
P_i(x) = \sigma_{ij} n_j = c_{ijkl} \frac{\partial u_k}{\partial x_l} n_j. \tag{3.3}
\]

For all points in \( \Omega \), the coefficients \( c_{ijkl}(x) \) are required to satisfy the usual symmetry condition

\[
c_{ijkl} = c_{ijlk} = c_{jkil} = c_{klij} \tag{3.4}
\]

and also sufficient conditions for the strain energy density to be positive. This requirement ensures that the system of partial differential equations is elliptic throughout \( \Omega \).
The coefficients in (3.1) are required to take the form

\[ c_{ijkl}(x) = c^{(0)}_{ijkl} g(x), \]

(3.5)

where the \( c^{(0)}_{ijkl} \) are constants and \( g(x_1, x_2, x_3) \) is a positive twice differentiable function of the variables \( x_1, x_2 \) and \( x_3 \) throughout \( \Omega \). Also in addition to the symmetry condition (3.4), the \( c^{(0)}_{ijkl} \) are required to satisfy the additional condition

\[ c^{(0)}_{ijkl} = c^{(0)}_{klij}. \]

(3.6)

Equation (3.1) may now be written in the form

\[ c^{(0)}_{ijkl} \frac{\partial}{\partial x_j} \left( g \frac{\partial u_k}{\partial x_l} \right) = 0. \]

(3.7)

Following Azis and Clements (11), consider a transformation of the dependent variables in the form

\[ u_k = g^{-1/2} \psi_k. \]

(3.8)

Use of (3.8) in (3.7) provides the equation

\[ c^{(0)}_{ijkl} \left[ g^{1/2} \frac{\partial^2 \psi_k}{\partial x_j \partial x_l} + g^{1/2} \frac{\partial \psi_k}{\partial x_j} \frac{\partial}{\partial x_l} - g^{1/2} \frac{\partial g^{1/2}}{\partial x_j} \frac{\partial \psi_k}{\partial x_l} - \psi_k \frac{\partial g^{1/2}}{\partial x_j} \frac{\partial g^{1/2}}{\partial x_l} \right] = 0, \]

(3.9)

where by virtue of (3.6), this equation reduces to

\[ g^{1/2} c^{(0)}_{ijkl} \frac{\partial^2 \psi_k}{\partial x_j \partial x_l} - \psi_k c^{(0)}_{ijkl} \frac{\partial^2 g^{1/2}}{\partial x_j \partial x_l} = 0. \]

(3.10)

Thus, if

\[ c^{(0)}_{ijkl} \frac{\partial^2 \psi_k}{\partial x_j \partial x_l} = 0 \]

(3.11)

and

\[ c^{(0)}_{ijkl} \frac{\partial^2 g^{1/2}}{\partial x_j \partial x_l} = 0, \]

(3.12)

then (3.10) will be satisfied. Thus, when \( g \) satisfies the system (3.12), the transformation given by (3.8) transforms the linear system with variable coefficients (3.7) to the linear system with constant coefficients (3.11).

As a result of the symmetry property \( c_{ijkl} = c_{klij} \), (3.12) consists of a system of six constant coefficients partial differential equations in the one dependent variable \( g^{1/2} \). In general, this system will be satisfied by a linear function of the three independent variables \( x_1, x_2 \) and \( x_3 \). Thus, \( g(x) \) may be taken in the form

\[ g(x) = (\alpha x_1 + \beta x_2 + \delta x_3 + \gamma)^2, \]

(3.13)

where \( \alpha, \beta, \delta \) and \( \gamma \) are constants that may be used to fit the analytical forms \( c_{ijkl}(x) = c^{(0)}_{ijkl} g(x) \) to given numerical values for the elastic moduli. Alternatively, the constants in (3.13) may be chosen
in the design of functionally graded materials that have desired properties regarding the stress and displacement in particular applications.

For the problems defined in section 2, the region under consideration Ω is either \( x_3 > 0 \) or \( 0 < x_3 < h \). Thus, in order to satisfy the condition that the \( g(x) \) given by (3.13) be positive throughout Ω, it is sufficient to require that \( \alpha = \beta = 0, \delta \geq 0 \) and \( \gamma > 0 \). Hence, (3.13) reduces to

\[
g(x) = (\delta x_3 + \gamma)^2. \tag{3.14}
\]

Now substitution of (3.5) and (3.8) into (3.3) yields

\[
P_l = -P_{ik}^{[g]} \psi_k + P_i^{[\psi]} g^{1/2}, \tag{3.15}
\]

where

\[
P_{ik}^{[g]}(x) = c_{ijkl}^{(0)} \frac{\partial g^{1/2}}{\partial x_l} n_j, \quad P_i^{[\psi]}(x) = c_{ijkl}^{(0)} \frac{\partial \psi_k}{\partial x_l} n_j. \tag{3.16}
\]

The stress is given by

\[
\sigma_{ij} = -c_{ijkl}^{(0)} \psi_k \frac{\partial g^{1/2}}{\partial x_l} + c_{ijkl}^{(0)} g^{1/2} \frac{\partial \psi_k}{\partial x_l}. \tag{3.17}
\]

Of particular interest in areas such as geomechanics is the case when the anisotropic half-space is transversely isotropic with the \( Ox_3 \) axis normal to the transverse planes and with the elastic moduli varying with the perpendicular distance from the plane boundary. Restricting attention to this case, the function \( g(x) \) is taken in the form given by (3.14). Also the non-zero \( c_{ijkl}^{(0)} \) may be expressed in terms of five constants \( A, N, F, C \) and \( L \) as follows:

\[
c_{1111}^{(0)} = c_{2222}^{(0)} = A, \quad c_{1122}^{(0)} = N, \quad c_{1133}^{(0)} = c_{2233}^{(0)} = F, \tag{3.18}
\]

\[
c_{1313}^{(0)} = c_{2323}^{(0)} = L, \quad c_{1212}^{(0)} = (A - N)/2, \quad c_{3333}^{(0)} = C. \tag{3.19}
\]

Use of (3.18) and (3.19) in (3.11) provides

\[
A \frac{\partial^2 \psi_1}{\partial x_1^2} + \frac{A - N}{2} \frac{\partial^2 \psi_1}{\partial x_2^2} + L \frac{\partial^2 \psi_1}{\partial x_3^2} + \frac{\partial}{\partial x_1} \left[ A + N \frac{\partial \psi_2}{\partial x_2} + (F + L) \frac{\partial \psi_3}{\partial x_3} \right] = 0, \tag{3.20}
\]

\[
A \frac{\partial^2 \psi_2}{\partial x_1^2} + \frac{A - N}{2} \frac{\partial^2 \psi_2}{\partial x_2^2} + L \frac{\partial^2 \psi_2}{\partial x_3^2} + \frac{\partial}{\partial x_2} \left[ A + N \frac{\partial \psi_1}{\partial x_1} + (F + L) \frac{\partial \psi_3}{\partial x_3} \right] = 0, \tag{3.21}
\]

\[
L \left[ \frac{\partial^2 \psi_3}{\partial x_1^2} + \frac{\partial^2 \psi_3}{\partial x_2^2} \right] + C \frac{\partial^2 \psi_3}{\partial x_3^2} + (F + L) \frac{\partial}{\partial x_3} \left[ \frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2} \right] = 0. \tag{3.22}
\]

Following Elliott (12), we assume a solution in the form

\[
\psi_1 = \frac{\partial \phi}{\partial x_1}, \quad \psi_2 = \frac{\partial \phi}{\partial x_2}, \quad \psi_3 = k \frac{\partial \phi}{\partial x_3}. \tag{3.23}
\]

Equations (3.20) and (3.21) will be satisfied if

\[
A \left[ \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} \right] + \left[ L + k(F + L) \right] \frac{\partial^2 \phi}{\partial x_3^2} = 0, \tag{3.24}
\]
and (3.22) is satisfied if
\[
[(F + L) + kL] \left[ \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} \right] + kC \frac{\partial^2 \phi}{\partial x_3^2} = 0. \tag{3.25}
\]

Set
\[
\frac{L + k(F + L)}{A} = \frac{kC}{(F + L) + kL} = \nu. \tag{3.26}
\]
Elimination of \( k \) from (3.26) yields the quadratic for \( \nu \)
\[
AL\nu^2 + [F(F + 2L) − AC]\nu + CL = 0. \tag{3.27}
\]
Thus, (3.24) and (3.25) will be satisfied if \( \phi \) is given by
\[
\left[ \Delta^2 + \nu_\alpha \frac{\partial^2}{\partial x_\alpha^2} \right] \phi_\alpha = 0 \quad \text{for} \quad \alpha = 1, 2, \tag{3.28}
\]
where \( \Delta^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \) and \( \nu_1, \nu_2 \) are the roots of the quadratic (3.27). Equation (3.23) thus yields
\[
\psi_1 = \frac{\partial}{\partial x_1}(\phi_1 + \phi_2), \quad \psi_2 = \frac{\partial}{\partial x_2}(\phi_1 + \phi_2), \quad \psi_3 = k_1 \frac{\partial \phi_1}{\partial x_3} + k_2 \frac{\partial \phi_2}{\partial x_3}. \tag{3.29}
\]
Substitute (3.29) into (3.8) and (3.17) to obtain the displacements
\[
u_1 = \frac{\partial}{\partial x_1}(\phi_1 + \phi_2), \quad \nu_2 = \frac{\partial}{\partial x_2}(\phi_1 + \phi_2), \quad \nu_3 = k_1 \frac{\partial \phi_1}{\partial x_3} + k_2 \frac{\partial \phi_2}{\partial x_3}. \tag{3.29}
\]
and the stresses are
\[
\sigma_{11} = g^{1/2} \left[ A \frac{\partial^2}{\partial x_1^2}(\phi_1 + \phi_2) + N \frac{\partial^2}{\partial x_2^2}(\phi_1 + \phi_2) + F \left( k_1 \frac{\partial^2 \phi_1}{\partial x_3^2} + k_2 \frac{\partial^2 \phi_2}{\partial x_3^2} \right) \right] - F \delta \left( k_1 \frac{\partial \phi_1}{\partial x_3} + k_2 \frac{\partial \phi_2}{\partial x_3} \right), \tag{3.32}
\]
\[
\sigma_{22} = g^{1/2} \left[ A \frac{\partial^2}{\partial x_1^2}(\phi_1 + \phi_2) + N \frac{\partial^2}{\partial x_2^2}(\phi_1 + \phi_2) + F \left( k_1 \frac{\partial^2 \phi_1}{\partial x_3^2} + k_2 \frac{\partial^2 \phi_2}{\partial x_3^2} \right) \right] - F \delta \left( k_1 \frac{\partial \phi_1}{\partial x_3} + k_2 \frac{\partial \phi_2}{\partial x_3} \right), \tag{3.33}
\]
\[
\sigma_{33} = g^{1/2} \left[ F \frac{\partial^2}{\partial x_1^2} (\phi_1 + \phi_2) + F \frac{\partial^2}{\partial x_2^2} (\phi_1 + \phi_2) + C \left( k_1 \frac{\partial^2 \phi_1}{\partial x_3^2} + k_2 \frac{\partial^2 \phi_2}{\partial x_3^2} \right) \right] \\
- C \delta \left( k_1 \frac{\partial \phi_1}{\partial x_3} + k_2 \frac{\partial \phi_2}{\partial x_3} \right),
\]
(3.34)

\[
\sigma_{12} = g^{1/2} (A - N) \frac{\partial^2}{\partial x_1 x_2} (\phi_1 + \phi_2),
\]
(3.35)

\[
\sigma_{\alpha 3} = g^{1/2} L \left[ (1 + k_1) \frac{\partial^2 \phi_1}{\partial x_\alpha x_3} + (1 + k_2) \frac{\partial^2 \phi_2}{\partial x_\alpha x_3} \right] - L \delta \frac{\partial}{\partial x_\alpha} (\phi_1 + \phi_2), \quad \alpha = 1, 2.
\]
(3.36)

In terms of cylindrical coordinates \((r, \theta, z)\), (3.28) can be written as

\[
\left[ \frac{\partial^2}{\partial r^2} + 1 \frac{\partial}{r \partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \nu_a \frac{\partial^2}{\partial z^2} \right] \phi_\alpha = 0 \quad \text{for} \quad \alpha = 1, 2.
\]
(3.37)

Also the displacements in cylindrical coordinates are

\[
u_r = g^{-\frac{1}{2}} \frac{\partial}{\partial r} (\phi_1 + \phi_2), \quad u_\theta = g^{-\frac{1}{2}} \frac{1}{r} \frac{\partial}{\partial \theta} (\phi_1 + \phi_2), \quad u_z = g^{-\frac{1}{2}} (k_1 \frac{\partial \phi_1}{\partial z} + k_2 \frac{\partial \phi_2}{\partial z}),
\]
(3.38)

and the stresses are

\[
\sigma_{rr} = g^{1/2} \left[ A \frac{\partial^2}{\partial r^2} (\phi_1 + \phi_2) + \frac{N}{r} \frac{\partial}{\partial r} (\phi_1 + \phi_2) + \frac{N}{r^2} \frac{\partial^2}{\partial \theta^2} (\phi_1 + \phi_2) \right] \\
+ F \left( k_1 \frac{\partial^2 \phi_1}{\partial z^2} + k_2 \frac{\partial^2 \phi_2}{\partial z^2} \right) - F \delta \left( k_1 \frac{\partial \phi_1}{\partial z} + k_2 \frac{\partial \phi_2}{\partial z} \right),
\]
(3.39)

\[
\sigma_{\theta \theta} = g^{1/2} \left[ N \frac{\partial^2}{\partial r^2} (\phi_1 + \phi_2) + A \frac{\partial}{\partial r} (\phi_1 + \phi_2) + A \frac{\partial^2}{\partial \theta^2} (\phi_1 + \phi_2) \right] \\
+ F \left( k_1 \frac{\partial^2 \phi_1}{\partial z^2} + k_2 \frac{\partial^2 \phi_2}{\partial z^2} \right) - F \delta \left( k_1 \frac{\partial \phi_1}{\partial z} + k_2 \frac{\partial \phi_2}{\partial z} \right),
\]
(3.40)

\[
\sigma_{zz} = g^{1/2} \left[ F \frac{\partial^2}{\partial r^2} (\phi_1 + \phi_2) + \frac{F}{r} \frac{\partial}{\partial r} (\phi_1 + \phi_2) + \frac{F}{r^2} \frac{\partial^2}{\partial \theta^2} (\phi_1 + \phi_2) \right] \\
+ C \left( k_1 \frac{\partial^2 \phi_1}{\partial z^2} + k_2 \frac{\partial^2 \phi_2}{\partial z^2} \right) - C \delta \left( k_1 \frac{\partial \phi_1}{\partial z} + k_2 \frac{\partial \phi_2}{\partial z} \right),
\]
(3.41)

\[
\sigma_{r \theta} = g^{1/2} (A - N) \left[ \frac{1}{r} \frac{\partial^2}{\partial r \partial \theta} (\phi_1 + \phi_2) - \frac{1}{r^2} \frac{\partial}{\partial \theta} (\phi_1 + \phi_2) \right],
\]
(3.42)

\[
\sigma_{rz} = g^{1/2} \left[ (1 + k_1) L \frac{\partial^2 \phi_1}{\partial r \partial z} + (1 + k_2) L \frac{\partial^2 \phi_2}{\partial r \partial z} \right] - L \delta \frac{\partial}{\partial r} (\phi_1 + \phi_2),
\]
(3.43)
\[ \sigma_{\theta z} = g^{1/2} \left[ (1 + k_1)L \frac{\partial^2 \phi_1}{\partial \theta \partial z} + (1 + k_2)L \frac{\partial^2 \phi_2}{\partial \theta \partial z} \right] - \frac{L \delta}{r} \frac{\partial}{\partial \theta} (\phi_1 + \phi_2). \] (3.44)

Attention is now restricted to the axially symmetric case, so that \( \phi_1 \) and \( \phi_2 \) are independent of \( \theta \). Hence, (3.37) reduces to

\[ \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + v_\alpha \frac{\partial^2}{\partial z^2} \right] \phi_\alpha = 0 \text{ for } \alpha = 1, 2, \] (3.45)

the non-zero displacements are \( u_r \) and \( u_z \) (given by (3.38)) and the non-zero stresses are \( \sigma_{rr}, \sigma_{\theta \theta}, \sigma_{zz} \) and \( \sigma_{rz} \) (given by (3.39)–(3.44) with \( \partial/\partial \theta \equiv 0 \)).

4. A half-space with specified axially symmetric boundary force

Let the boundary \( z = 0 \) of the inhomogeneous elastic medium \( z \geq 0 \) be subjected to the axisymmetric pressure \( p(r) \) acting normal to the boundary, so that the boundary conditions on the plane \( z = 0 \) are given by

\[ \sigma_{zz} = -p(r), \quad \sigma_{rz} = 0. \] (4.1)

Multiplying both sides of (3.45) by \( r J_0(\zeta r) \) and integrating from 0 to \( \infty \) yields

\[ \int_0^\infty r \left( \frac{\partial^2 \phi_\alpha}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_\alpha}{\partial r} + v_\alpha \frac{\partial^2 \phi_\alpha}{\partial z^2} \right) J_0(\zeta r) \, dr + \int_0^\infty r v_\alpha \frac{\partial^2 \phi_\alpha}{\partial z^2} J_0(\zeta r) \, dr = 0. \] (4.2)

Proceeding further as in Sneddon (13), we obtain

\[ -\zeta^2 \tilde{\phi}_\alpha(\zeta, z) + v_\alpha \int_0^\infty r \frac{d^2 \phi_\alpha}{dz^2} J_0(\zeta r) \, dr = 0, \] (4.3)

or

\[ \frac{d^2}{dz^2} \tilde{\phi}_\alpha(\zeta, z) - \frac{\zeta^2}{v_\alpha} \tilde{\phi}_\alpha(\zeta, z) = 0, \] (4.4)

where

\[ \tilde{\phi}_\alpha(\zeta, z) = \int_0^\infty r \phi_\alpha(r) J_0(\zeta r) \, dr. \] (4.5)

Equation (4.4) has solution

\[ \tilde{\phi}_\alpha(\zeta, z) = A_\alpha(\zeta)e^{-\zeta z_\alpha} + B_\alpha(\zeta)e^{\zeta z_\alpha}, \] (4.6)

where \( z_\alpha = z/\sqrt{\nu_\alpha} \). For the problem under consideration, the material occupies the region \( z > 0 \) and since the term \( B_\alpha(\zeta)e^{\zeta z_\alpha} \) in (4.6) would give rise to infinite displacement and stress as \( z \to +\infty \), we take \( B_\alpha(\zeta) = 0 \) and hence restrict \( \phi \) to the form

\[ \tilde{\phi}_\alpha(\zeta, z) = A_\alpha(\zeta)e^{-\zeta z_\alpha}, \] (4.7)

where in the case of complex values for \( \nu_\alpha \), the positive real part of \( \sqrt{\nu_\alpha} \) will be taken. Transforming back as in Sneddon (13) provides

\[ \phi_\alpha(r, z) = \int_0^\infty \zeta A_\alpha(\zeta)e^{-\zeta z_\alpha} J_0(\zeta r) d\zeta = \int_0^\infty \zeta \tilde{\phi}_\alpha(\zeta, z) J_0(\zeta r) d\zeta. \] (4.8)
Use of (4.8) in (3.38)–(3.44) yields the non-zero displacements and stresses in integral form,

\[ u_r = -g^{-\frac{1}{2}} \int_0^\infty \xi^2 (\bar{\phi}_1 + \bar{\phi}_2) J_1(\xi r) d\xi, \]  
(4.9)

\[ u_\zeta = g^{-\frac{1}{2}} \int_0^\infty \xi \left( k_1 \frac{d \bar{\phi}_1}{dz} + k_2 \frac{d \bar{\phi}_2}{dz} \right) J_0(\xi r) d\xi, \]  
(4.10)

\[ \sigma_{rr} = \int_0^\infty \xi J_0(\xi r) \left\{ -A \xi^2 g^\frac{1}{2} (\bar{\phi}_1 + \bar{\phi}_2) + F \left( k_1 \frac{d^2 \bar{\phi}_1}{dz^2} + k_2 \frac{d^2 \bar{\phi}_2}{dz^2} \right) \right\} d\xi, \]  
(4.11)

\[ \sigma_{\theta\theta} = \int_0^\infty \xi J_0(\xi r) \left\{ -N \xi^2 g^\frac{1}{2} (\bar{\phi}_1 + \bar{\phi}_2) + F \left( k_1 \frac{d^2 \bar{\phi}_1}{dz^2} + k_2 \frac{d^2 \bar{\phi}_2}{dz^2} \right) \right\} d\xi, \]  
(4.12)

\[ \sigma_{zz} = \int_0^\infty \xi J_0(\xi r) \left\{ -F \xi^2 g^\frac{1}{2} (\bar{\phi}_1 + \bar{\phi}_2) + C g^\frac{1}{2} \left( k_1 \frac{d^2 \bar{\phi}_1}{dz^2} + k_2 \frac{d^2 \bar{\phi}_2}{dz^2} \right) \right\} d\xi, \]  
(4.13)

\[ \sigma_{r\zeta} = \int_0^\infty L \xi^2 J_1(\xi r) \left\{ -g^\frac{1}{2} \left[ (1 + k_1) \frac{d \bar{\phi}_1}{dz} + (1 + k_2) \frac{d \bar{\phi}_2}{dz} \right] + \delta(\bar{\phi}_1 + \bar{\phi}_2) \right\} d\xi. \]  
(4.14)

From (4.8), (4.13) and (4.14), it follows that the boundary conditions (4.1) will be satisfied if \( A_1(\bar{\zeta}) \) and \( A_2(\bar{\zeta}) \) satisfy the equations

\[ \frac{-\bar{p}(\zeta)}{\xi^2} = \left[ \kappa_1 + \lambda_1 \frac{\delta}{\zeta} \right] A_1(\bar{\zeta}) + \left[ \kappa_2 + \lambda_2 \frac{\delta}{\zeta} \right] A_2(\bar{\zeta}), \]  
(4.15)

\[ 0 = \left[ \eta_1 + \frac{\delta}{\zeta} \right] A_1(\bar{\zeta}) + \left[ \eta_2 + \frac{\delta}{\zeta} \right] A_2(\bar{\zeta}), \]  
(4.16)

where \( \kappa_a = (-F + Ck_a/\nu_a)\gamma \), \( \lambda_a = Ck_a/\sqrt{\nu_a} \), \( \eta_a = (1 + k_a)\gamma/\sqrt{\nu_a} \) and

\[ \bar{p}(\zeta) = \int_0^\infty r p(r) J_0(\xi r) dr. \]  
(4.17)

Equations (4.15) and (4.16) may be solved for \( A_1(\bar{\zeta}) \) and \( A_2(\bar{\zeta}) \) to yield

\[ A_1(\bar{\zeta}) = \frac{-\bar{p}(\eta_2 \xi + \delta)}{t_1 \xi^3 + t_2 \xi^2 + t_3 \xi}, \quad A_2(\bar{\zeta}) = \frac{\bar{p}(\eta_1 \xi + \delta)}{t_1 \xi^3 + t_2 \xi^2 + t_3 \xi}, \]  
(4.18)

where \( t_1 = \kappa_1 \eta_2 - \kappa_2 \eta_1, \quad t_2 = \delta(\kappa_1 - \kappa_2 + \lambda_1 \eta_2 - \lambda_2 \eta_1) \) and \( t_3 = \delta^2(\lambda_1 - \lambda_2) \).
Of interest is the case where the boundary \( z = 0 \) is subjected to a constant normal pressure \( P \) in the region \( r < a \), where \( a \) is a positive constant and for \( r > a \), the boundary surface is traction free. In this case, the function \( p(r) \) in (4.1) becomes

\[
p(r) = \begin{cases} 
P & \text{for } 0 \leq r < a, \\
0 & \text{for } r > a.
\end{cases}
\]

(4.19)

Transforming this condition using the Hankel transform yields

\[
\tilde{p}(\xi) = (aP/\xi) J_1(a\xi).
\]

(4.20)

Substituting (4.18) and (4.20) into (4.7), the transforms \( \tilde{\phi}_1 \) and \( \tilde{\phi}_2 \) can now be written in the form

\[
\tilde{\phi}_1 = \frac{-aP[\eta_2 \xi + \delta]J_1(a\xi)}{\xi^2[t_1 \xi^2 + t_2 \xi + t_3]} e^{-\xi z_1}, \quad \tilde{\phi}_2 = \frac{aP[\eta_1 \xi + \delta]J_1(a\xi)}{\xi^2[t_1 \xi^2 + t_2 \xi + t_3]} e^{-\xi z_2}.
\]

(4.21)

Using these, (4.9)–(4.14) become

\[
u_r = g r^{1/2} \int_0^\infty \frac{aP J_1(\xi r) J_1(\xi a)}{(t_1 \xi^2 + t_2 \xi + t_3)} \left\{ (\eta_2 \xi + \delta) e^{-\xi z_1} - (\eta_1 \xi + \delta) e^{-\xi z_2} \right\} d\xi,
\]

(4.22)

\[
u_z = g r^{1/2} \int_0^\infty \frac{aP J_0(\xi r) J_1(\xi a)}{(t_1 \xi^2 + t_2 \xi + t_3)} \left\{ \frac{k_1}{\sqrt{v_1}} (\eta_2 \xi + \delta) e^{-\xi z_1} - \frac{k_2}{\sqrt{v_2}} (\eta_1 \xi + \delta) e^{-\xi z_2} \right\} d\xi,
\]

(4.23)

\[
\sigma_{rr} = \int_0^\infty \left[ \frac{aP \xi J_0(\xi r) J_1(\xi a)}{(t_1 \xi^2 + t_2 \xi + t_3)} \left\{ (\eta_2 \xi + \delta) \left( A(\delta z + \gamma) - \frac{Fk_1}{v_1} - \frac{Fk_1}{\sqrt{v_1}} \right) e^{-\xi z_1}
\right.ight.

- \left. (\eta_1 \xi + \delta) \left( A(\delta z + \gamma) - \frac{Fk_2}{v_2} - \frac{Fk_2}{\sqrt{v_2}} \right) e^{-\xi z_2} \right\}

\left. + \frac{aP J_1(\xi r) J_1(\xi a)}{(t_1 \xi^2 + t_2 \xi + t_3)} \frac{N - A}{r} \left\{ (\eta_2 \xi + \delta) e^{-\xi z_1} - (\eta_1 \xi + \delta) e^{-\xi z_2} \right\} \right] d\xi,
\]

(4.24)

\[
\sigma_{\theta\theta} = \int_0^\infty \left[ \frac{aP \xi J_0(\xi r) J_1(\xi a)}{(t_1 \xi^2 + t_2 \xi + t_3)} \left\{ (\eta_2 \xi + \delta) \left( N(\delta z + \gamma) - \frac{Fk_1}{v_1} - \frac{Fk_1}{\sqrt{v_1}} \right) e^{-\xi z_1}
\right.ight.

- \left. (\eta_1 \xi + \delta) \left( N(\delta z + \gamma) - \frac{Fk_2}{v_2} - \frac{Fk_2}{\sqrt{v_2}} \right) e^{-\xi z_2} \right\}

\left. + \frac{aP J_1(\xi r) J_1(\xi a)}{(t_1 \xi^2 + t_2 \xi + t_3)} \frac{A - N}{r} \left\{ (\eta_2 \xi + \delta) e^{-\xi z_1} - (\eta_1 \xi + \delta) e^{-\xi z_2} \right\} \right] d\xi,
\]

(4.25)

\[
\sigma_{zz} = \int_0^\infty \frac{-aP J_0(\xi r) J_1(\xi a)}{(t_1 \xi^2 + t_2 \xi + t_3)} \left\{ (\eta_2 \xi + \delta) \left[ -F(\delta z + \gamma) + \frac{C(\delta z + \gamma)k_1}{v_1} \right] e^{-\xi z_1}
\right.

- \left. (\eta_1 \xi + \delta) \left[ -F(\delta z + \gamma) + \frac{C(\delta z + \gamma)k_2}{v_2} \right] e^{-\xi z_2} \right\} d\xi,
\]

(4.26)
\[ \sigma_{rz} = \int_0^\infty \frac{-a P L J_1(\xi \tau) J_1(\xi a)}{(t_1 \xi^2 + t_2 \xi + t_3)} \left\{ (\eta_2 \xi + \delta) \left[ \frac{(\delta \xi + \gamma)(1 + k_1)\xi}{\sqrt{v_1}} + \delta \right] e^{-\xi z_1} \right. \\
\left. - (\eta_1 \xi + \delta) \left[ \frac{(\delta \xi + \gamma)(1 + k_2)\xi}{\sqrt{v_2}} + \delta \right] e^{-\xi z_2} \right\} d\xi, \quad (4.27) \]

Using (4.23) and after rearranging the constants, the displacement on the surface \( z = 0 \) can be written
\[ u_z = \int_0^\infty a P J_0(\xi \tau) J_1(\xi a) (s_4 \gamma \xi + s_5 \delta) d\xi, \quad (4.28) \]

where
\[ s_1 = \frac{1 + k_2}{\sqrt{v_2}} \left[ -F + \frac{C k_1}{v_1} \right] - \frac{1 + k_1}{\sqrt{v_1}} \left[ -F + \frac{C k_2}{v_2} \right], \quad s_2 = C \left[ \frac{k_1 - k_2}{v_1 v_2} \right], \quad (4.29) \]
\[ s_3 = C \left[ \frac{k_1}{\sqrt{v_1}} - \frac{k_2}{\sqrt{v_2}} \right], \quad s_4 = \frac{k_1 - k_2}{v_1 v_2}, \quad s_5 = \frac{k_1}{\sqrt{v_1}} - \frac{k_2}{\sqrt{v_2}}. \quad (4.30) \]

Let
\[ r' = r/a, \quad \zeta' = z/a, \quad \xi' = a \xi, \quad \delta' = a \delta, \quad (4.31) \]
\[ s_1' = s_1/S, \quad s_2' = s_2/S, \quad s_3' = s_3/S, \quad P' = P/S, \quad C' = C/S, \quad (4.32) \]

where \( S \) is a reference stress. In terms of these quantities, (4.28) may be written in the non-dimensional form
\[ \frac{u_z}{a} = \frac{P' s_5}{\gamma \delta' S_3} \int_0^\infty J_0(\xi' r') J_1(\xi') f(\xi') d\xi', \quad (4.33) \]

where
\[ f(\xi') = \frac{1 + (s_4/s_5)(\gamma/\delta')\xi'}{1 + (s_4'/s_5')(\gamma/\delta')\xi' + (s_3'/s_3')(\gamma/\delta')^2 \xi'^2}. \]

In order to examine the surface displacement when the half-space has small elastic moduli \( c_{ijkl}/c_{ijkl}^{(0)} = (\delta' \zeta' + \gamma)^2 = \delta'^2 (\zeta' + \gamma/\delta')^2 \) near the boundary \( \zeta' = 0 \), we consider the case when the ratio \( \gamma/\delta' \) is small. Using (4.30) and (4.32), (4.33) may be written in the form
\[ \frac{u_z}{a} = \frac{P'}{C' \gamma \delta'} \int_0^\infty J_0(\xi' r') J_1(\xi') f(\xi') d\xi' + \frac{P'}{C' \gamma \delta'} \int_{\sqrt{\delta'/\gamma}}^\infty J_0(\xi' r') J_1(\xi') f(\xi') d\xi'. \quad (4.34) \]

The integrand in the second integral in (4.34) is bounded in the interval \([\sqrt{\delta'/\gamma}, \infty)\) and hence this integral becomes small as \( \gamma/\delta' \) becomes small. Also in the first integral in (4.34), \( 0 \leq \xi' \leq \sqrt{\delta'/\gamma} \) and hence for \( \gamma/\delta' \) sufficiently small, the normal displacement \( u_z \) will be approximately given by
\[ \frac{u_z}{a} \simeq \frac{P'}{C' \gamma \delta'} \int_0^\infty J_0(\xi' r') J_1(\xi') d\xi' = \begin{cases} \frac{P'/(\gamma \delta' C')}{P'} & \text{for } 0 \leq r' < 1, \\
0 & \text{for } r' > 1, \end{cases} \quad (4.35) \]
using the Weber–Schafheitlin integral,

\[
\int_0^\infty J_0(\xi' r') J_1(\xi') \, d\xi' = \begin{cases} 
1 & \text{for } 0 \leq r' < 1, \\
0 & \text{for } r' > 1.
\end{cases}
\] (4.36)

Using (4.1), (4.19) and (4.32), the prescribed normal stress \(\sigma_{zz}\) on the boundary \(z' = 0\) may be written in the form

\[
\sigma_{zz} / S = \begin{cases} 
-P' & \text{for } 0 \leq r' < 1, \\
0 & \text{for } r' > 1.
\end{cases}
\] (4.37)

It follows from (4.35) and (4.37) that for \(\gamma / \delta'\) sufficiently small, the displacement will be approximately zero outside the contact region and over the contact region, it will be approximately given by a multiple of the applied stress \(\sigma_{zz} / S\). Also for a fixed and finite \(\gamma\), the surface displacement decreases as \(\delta'\) increases.

In view of the factor \(\gamma\) in the denominator in (4.28), it follows that, in general, the surface displacement increases as \(\gamma\) becomes small. Here, we examine a particular class of materials for which the displacement remains constant as \(\gamma\) becomes small. Let

\[
m = \gamma \delta',
\] (4.38)

where \(m\) is a finite constant. It follows from (4.35) and (4.37) that if \(m\) is held fixed, then as \(\gamma\) becomes small, the displacement is approximately zero outside the contact region and over the contact region, it is approximately equal to a constant finite multiple of the applied stress \(\sigma_{zz} / S\). This behaviour of the boundary displacement has some similarity to the behaviour of the displacement noted by Gibson (1) in a related problem for an incompressible homogeneous isotropic material.

5. An elastic layer on a rigid foundation

In this section, an axisymmetric contact problem of an inhomogeneous elastic layer that adheres to a rigid foundation is considered. The layer occupies the region \(0 < z < h\). On the boundary \(z = 0\), a normal stress \(\sigma_{zz}\) is applied so the boundary conditions on \(z = 0\) and the interface conditions on \(z = h\) are

\[
\begin{align*}
\sigma_{zz} &= -p(r), \quad \sigma_{rz} = 0, \quad \text{for } z = 0, \\
u_z &= 0, \quad u_r = 0, \quad \text{for } z = h.
\end{align*}
\] (5.1) (5.2)

For this problem, the stress and displacement are given by (4.9)–(4.14) with the function \(\tilde{\phi}_a(\xi, z)\) given by (4.6). Using (4.9)–(4.14) together with (4.6), it follows that the conditions (5.1) and (5.2) will be satisfied if the functions \(A_\alpha(\xi)\) and \(B_\alpha(\xi)\) satisfy a system of four linear equations which in matrix notation are given by

\[
\begin{pmatrix}
\kappa_1 + \lambda_1 \delta / \xi \\
\eta_1 + \delta / \xi \\
-\lambda_1 / C e^{-h_1 \xi} \\
e^{-h_1 \xi}
\end{pmatrix}
\begin{pmatrix}
A_1 \\
B_1 \\
A_2 \\
B_2
\end{pmatrix}
= 
\begin{pmatrix}
-\tilde{p} / \xi^2 \\
0 \\
0 \\
0
\end{pmatrix},
\] (5.3)
where $\tilde{p}$ is given by (4.17) and $h_\alpha = h/\sqrt{\alpha}$ for $\alpha = 1, 2$. The solution of this system is

$$A_1 = -\tilde{p}[\lambda_1 + \lambda_2)(-\eta_2 \xi + \delta)e^{2h_1 \xi} - (\lambda_1 - \lambda_2)(\eta_2 \xi + \delta)e^{2(h_1 + h_2) \xi}$$

$$+ 2\lambda_2(\eta_1 \xi - \delta)e^{(h_1 + h_2) \xi}] / D, \quad (5.4)$$

$$B_1 = \tilde{p}[(\lambda_1 + \lambda_2)(\eta_2 \xi + \delta)e^{2h_2 \xi} - 2\lambda_2(\eta_1 \xi + \delta)e^{(h_1 + h_2) \xi} + (\lambda_1 - \lambda_2)(\eta_2 \xi - \delta)] / D, \quad (5.5)$$

$$A_2 = -\tilde{p}[(\lambda_1 + \lambda_2)(-\eta_1 \xi + \delta)e^{2h_2 \xi} + (\lambda_1 - \lambda_2)(\eta_1 \xi + \delta)e^{2(h_1 + h_2) \xi}$$

$$+ 2\lambda_1(\eta_2 \xi - \delta)e^{(h_1 + h_2) \xi}] / D, \quad (5.6)$$

$$B_2 = \tilde{p}[(\lambda_1 + \lambda_2)(\eta_1 \xi + \delta)e^{2h_1 \xi} - 2\lambda_1(\eta_2 \xi + \delta)e^{(h_1 + h_2) \xi} - (\lambda_1 - \lambda_2)(\eta_1 \xi - \delta)] / D, \quad (5.7)$$

where

$$D = \xi[e^{2h_1 \xi} \{-\xi^2(\lambda_1 + \lambda_2)(\kappa_1 \eta_2 + \kappa_2 \eta_1) - \xi \delta[\lambda_1^2 \eta_2 - \lambda_2^2 \eta_1]$$

$$- (\lambda_1 + \lambda_2)(\kappa_1 - \kappa_2) - \lambda_1 \lambda_2(\eta_1 - \eta_2)] + \delta^2(\lambda_1 + \lambda_2)^2\}$$

$$+ e^{2h_2 \xi} \{-\xi^2(\lambda_1 + \lambda_2)(\kappa_1 \eta_2 + \kappa_2 \eta_1) + \xi \delta[\lambda_1^2 \eta_2 - \lambda_2^2 \eta_1]$$

$$- (\lambda_1 + \lambda_2)(\kappa_1 - \kappa_2) - \lambda_1 \lambda_2(\eta_1 - \eta_2)] + \delta^2(\lambda_1 + \lambda_2)^2\}$$

$$+ e^{2(h_1 + h_2) \xi} \{-\xi^2(\lambda_1 - \lambda_2)(\kappa_1 \eta_2 - \kappa_2 \eta_1) - \xi \delta[\lambda_1^2 \eta_2 + \lambda_2^2 \eta_1]$$

$$+ (\lambda_1 - \lambda_2)(\kappa_1 - \kappa_2) - \lambda_1 \lambda_2(\eta_1 + \eta_2)] - \delta^2(\lambda_1 - \lambda_2)^2\}$$

$$+ 4e^{(h_1 + h_2) \xi} \{-\xi^2(\lambda_1 \kappa_2 \eta_2 + \lambda_2 \kappa_1 \eta_1) - 2\lambda_1 \lambda_2 \delta^2\}$$

$$+ \{-\xi^2(\lambda_1 - \lambda_2)(\kappa_1 \eta_2 - \kappa_2 \eta_1) + \xi \delta[\lambda_1^2 \eta_2 + \lambda_2^2 \eta_1]$$

$$+ (\lambda_1 - \lambda_2)(\kappa_1 - \kappa_2) - \lambda_1 \lambda_2(\eta_1 + \eta_2)] - \delta^2(\lambda_1 - \lambda_2)^2\}]. \quad (5.8)$$

These expressions for $A_\alpha(\xi)$ and $B_\alpha(\xi)$ in terms of known constants and the variable $\xi$ may be used in (4.6) and (4.9)–(4.14) to give integral forms for the stress and displacement from which numerical values may be calculated. In particular,

$$u_z = g^{-\frac{1}{2}} \int_0^\infty \hat{\xi} \left[ k_1 \left( -\xi \frac{A_1}{\sqrt{v_1}} e^{-\xi z_1} + \xi \frac{B_1}{\sqrt{v_1}} e^{\xi z_1} \right) 

+ k_2 \left( -\xi \frac{A_2}{\sqrt{v_2}} e^{-\xi z_2} + \xi \frac{B_2}{\sqrt{v_2}} e^{\xi z_2} \right) \right] J_0(\xi r) \, d\xi. \quad (5.9)$$

From (5.3), it is apparent that if the layer thickness $h$ is sufficiently large for terms of order $\exp(-\xi h_1)$ and $\exp(-\xi h_2)$ to be ignored, then the last two rows of the matrix provide two homogeneous linear algebraic equations for the two unknowns $B_1$ and $B_2$. Also the determinant of the matrix of coefficients of these two equations is given by $\exp(\xi h_1) \exp(\xi h_2)(\lambda_1 - \lambda_2)/C$ and since, in general, $\lambda_1$ and $\lambda_2$ are distinct, the determinant of coefficients is non-zero. Hence, $B_1$ and $B_2$ are both zero. The matrix equation therefore degenerates into the matrix form of the system of two equations (4.15) and (4.16). Hence, as $h$ becomes sufficiently large, the displacement and stress for the slab approach the displacement and stress for the corresponding problem for a half-space. Also it follows that for $h$ sufficiently large, the surface displacement for the layer as $\gamma$ becomes small will
exhibit similar behaviour to that described at the end of section 4 for the corresponding problem for the half-space.

6. An isotropic half-space and layer on a rigid foundation

For isotropic materials, the non-zero elastic coefficients $c_{ijkl}$ may be expressed in terms of the two Lamé coefficients $\lambda$ and $\mu$ in the form

$$\lambda = \lambda^{(0)} g(x), \quad \mu = \mu^{(0)} g(x),$$

where $\lambda^{(0)}$ and $\mu^{(0)}$ are constants. For an isotropic material, the relevant coefficients $c_{ijkl}$ are related to $\lambda$ and $\mu$ by the equations

$$c_{1111} = c_{2222} = c_{3333} = \lambda + 2\mu,$$  \hspace{1cm} (6.2)

$$c_{1122} = c_{1133} = c_{2233} = \lambda,$$  \hspace{1cm} (6.3)

$$c_{1212} = c_{1313} = c_{2323} = \mu,$$  \hspace{1cm} (6.4)

and the constants $c_{ijkl}^{(0)}$ and $A, N, F, C$ and $L$ may be expressed in terms of the two constants $\lambda^{(0)}$ and $\mu^{(0)}$ by the equations

$$c_{1111}^{(0)} = c_{2222}^{(0)} = c_{3333}^{(0)} = A = C = \lambda^{(0)} + 2\mu^{(0)},$$

$$c_{1122}^{(0)} = c_{1133}^{(0)} = c_{2233}^{(0)} = N = F = \lambda^{(0)},$$

$$c_{1212}^{(0)} = c_{1313}^{(0)} = c_{2323}^{(0)} = L = \mu^{(0)}.$$  \hspace{1cm} (6.5)

The coefficients $c_{ijkl}$ must satisfy the condition (3.6) which requires that $c_{ijkl} = c_{ilkj}$ so that from (6.3), (6.4), (6.6) and (6.7), it follows that

$$\lambda = \mu, \quad \lambda^{(0)} = \mu^{(0)}. \hspace{1cm} (6.8)$$

Young’s modulus $E$ and Poisson’s ratio $\nu$ are related to the Lamé coefficients by the equations

$$E = \frac{\mu (3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}. \hspace{1cm} (6.9)$$

Hence, by virtue of (6.8), it follows that

$$E = \frac{5\lambda}{2}, \quad \nu = \frac{1}{4}. \hspace{1cm} (6.10)$$

Thus, the analysis of the previous two sections may be applied to an inhomogeneous isotropic material for which the Poisson’s ratio $\nu$ is 1/4 (a frequently occurring value in rock materials (14)) and the inhomogeneity is specified by either Young’s modulus or one of the Lamé coefficients in the forms

$$E = E^{(0)}(\delta z + \gamma)^2, \quad \lambda = \lambda^{(0)}(\delta z + \gamma)^2, \quad \mu = \mu^{(0)}(\delta z + \gamma)^2,$$  \hspace{1cm} (6.11)

where these alternative forms are related by

$$E = 5\lambda/2, \quad \lambda = \mu, \hspace{1cm} (6.12)$$
so that
\[ E^{(0)} = 5\lambda^{(0)}/2, \quad \mu^{(0)} = \frac{\lambda^{(0)}}{2}. \quad (6.13) \]

For the isotropic case, the quadratic (3.27) has equal roots. As a consequence, the matrix of coefficients for the system (4.15) and (4.16) and the matrix (5.3) are singular and hence their inverses do not exist. Thus, for the axisymmetric problems, the analysis of the previous two sections is no longer valid. In order to employ the analysis of the previous sections to obtain numerical results for the isotropic case, the numerical values of the isotropic elastic moduli are perturbed slightly from their exact isotropic values in order to obtain distinct roots for the quadratic (3.27). Specifically, the relevant constants \( A, N, F, C \) and \( L \) are chosen in the form
\[ A = C = 3\lambda^{(0)}, \quad F = \lambda^{(0)} - \epsilon, \quad N = L = \lambda^{(0)}, \quad (6.14) \]
where \( \epsilon \) is a small positive parameter. As \( \epsilon \to 0 \), numerical values obtained from the relevant equations for the displacement and stress vector in the previous two sections tend to the values for an isotropic material with Poisson’s ratio 1/4.

7. The particular class of transversely isotropic materials

The analysis of the previous sections only applies to transversely isotropic materials for which the \( c_{ijkl}^{(0)} \) satisfy the conditions (3.6). For the non-zero \( c_{ijkl}^{(0)} \) given by (3.18) and (3.19), these conditions will be satisfied if \( c_{1133}^{(0)} = c_{1313}^{(0)} \) and \( c_{1122}^{(0)} = c_{1212}^{(0)} \) which in terms of the constants \( A, N, F, C \) and \( L \) gives
\[ N = A/3 \text{ and } F = L. \quad (7.1) \]
Thus, the current analysis applies to transversely isotropic materials for which the \( c_{ijkl}^{(0)} \) may be expressed in terms of the constant \( C \) and either one of the constants \( A \) and \( N \) together with either one of the constants \( F \) and \( L \).

In order to investigate the physical significance of the constraints (7.1), it is convenient at this point to introduce the following representation for the elastic coefficients for transversely isotropic materials commonly used in engineering and geomechanics applications:
\[ E_1 = E_1^{(0)}(\delta z + \gamma)^2, \quad E_2 = E_2^{(0)}(\delta z + \gamma)^2, \quad G = G^{(0)}(\delta z + \gamma)^2, \quad \nu_1, \nu_2. \quad (7.2) \]
In (7.2), \( E_1 \) and \( E_2 \) denote Young’s moduli in the transverse plane and in the direction normal to it, \( \nu_1 \) and \( \nu_2 \) denote Poisson’s ratios characterizing the lateral strain response in the transverse plane to a stress acting parallel or normal to it, respectively, and \( G \) is the shear modulus in planes normal to the transverse plane. Also \( E_1^{(0)}, E_2^{(0)}, \nu_1, \nu_2 \) and \( G^{(0)} \) are constants, which are related to \( A, N, F, C \) and \( L \) by
\[ E_1^{(0)} = D/(AC - F^2), \quad E_2^{(0)} = D/(A^2 - N^2), \quad G^{(0)} = L, \quad (7.3) \]
\[ \nu_1 = (NC - F^2)/(AC - F^2), \quad \nu_2 = F/(A + N), \quad (7.4) \]
where \( D = A^2C - 2AF^2 - N^2C + 2NF^2 \).
The non-zero \( c_{ijkl}^{(0)} \) may be expressed in terms of five elastic constants \( E_1^{(0)}, E_2^{(0)}, \nu_1, \nu_2 \) and \( G^{(0)} \) as follows (10):

\[
\begin{align*}
    c_{1111}^{(0)} &= c_{2222}^{(0)} = A = E_1^{(0)}(1 - E_1^{(0)}\nu_2^2/E_2^{(0)})/[(1 + \nu_1)\mathcal{D}], \\
    c_{1122}^{(0)} &= N = E_1^{(0)}(1 - E_1^{(0)}\nu_2^2/E_2^{(0)})/[(1 + \nu_1)\mathcal{D}] - E_1^{(0)}/(1 + \nu_1), \\
    c_{1133}^{(0)} &= c_{2233}^{(0)} = F = E_1^{(0)}\nu_2/\mathcal{D}, \\
    c_{1212}^{(0)} &= (A - N)/2 = E_1^{(0)}/[2(1 + \nu_1)],
\end{align*}
\]

where \( \mathcal{D} = 1 - \nu_1 - 2E_1^{(0)}\nu_2^2/E_2^{(0)} \). The constraints (7.1) on the constants \( A, N, F, C \) and \( L \) together with (7.5)–(7.8) lead to the following constraints on the constants \( E_1^{(0)}, E_2^{(0)}, \nu_1, \nu_2 \) and \( G^{(0)} \):

\[
\begin{align*}
    E_1^{(0)} &= 1 - 3\nu_1/4 \quad \text{and} \quad E_2^{(0)} = 1 + \nu_1/2 \nu_2.
\end{align*}
\]

Thus, the current analysis applies to transversely isotropic materials for which the non-zero \( c_{ijkl}^{(0)} \) may be expressed in terms of any three of the constants \( E_1^{(0)}, E_2^{(0)}, \nu_1, \nu_2 \) and \( G^{(0)} \).

For an isotropic material, \( \nu_1 = \nu_2 \) and \( E_1 = E_2 \). Let \( \nu_1 = \nu_2 = \nu \) and \( E_1 = E_2 = E \). The shear modulus \( G \) for isotropic materials is related to \( E \) and \( \nu \) by \( G = E/[2(1 + \nu)] \). Then, from (7.2) and (7.9), it follows that \( \nu = 1/4 \), \( E = 5G/2 \) and \( E^{(0)} = 5G^{(0)}/2 \) which is consistent with the results in (6.12) and (6.13) (with \( G = \nu \)).

This suggests that the class of transversely isotropic materials which satisfy the constraints (7.1) or alternatively (7.9) may be employed to closely model transversely isotropic materials which are not too strongly anisotropic and which have Poisson’s ratios in the vicinity of 0.25. Such materials are important in the area of rock mechanics with many intact rocks falling within this class. Specifically, the estimation of transversely isotropic intact rock properties indicates that a substantial number of such rocks are not too strongly anisotropic and have Poisson’s ratios between 0.2 and 0.3 (see, for example, (15) to (18)).

8. Numerical results

For the purposes of obtaining some numerical values for the normal displacement and stress, it is useful to employ the following non-dimensional quantities:

\[
\begin{align*}
    h' &= h/a, \\
    z_1' &= z_1/a, \\
    z_2' &= z_2/a, \\
    c_{ijkl}' &= c_{ijkl}^{(0)}/S, \\
    \lambda' &= \lambda^{(0)}/S, \\
    \epsilon' &= \epsilon/S, \\
    A' &= A/S, \\
    N' &= N/S, \\
    F' &= F/S, \\
    L' &= L/S, \\
    E_1' &= E_1^{(0)}/S, \\
    E_2' &= E_2^{(0)}/S, \\
    G' &= G^{(0)}/S, \\
    \xi_1' &= \xi_1/S, \\
    \xi_2' &= \xi_2/S, \\
    \kappa' &= \kappa/S, \\
    A_\alpha' &= A_\alpha/S, \\
    B_\alpha' &= B_\alpha/S,
\end{align*}
\]

where \( S \) is the reference stress. Referred to the non-dimensional quantities in (4.31), (4.32) and (8.1)–(8.4), (4.23), (4.26) and (5.9) become

\[
\begin{align*}
    \frac{u_z}{a} = \frac{1}{\gamma + \delta z'} \int_0^\infty P' J_0(\xi' z') J_1(\xi') \left\{ \frac{k_1}{\sqrt{\nu_1}} (\eta_2 \xi' + \delta') e^{-\xi' z} - \frac{k_2}{\sqrt{\nu_2}} (\eta_1 \xi' + \delta') e^{-\xi' z'} \right\} d\xi',
\end{align*}
\]
\[
\frac{\sigma_{zz}}{S} = \int_0^\infty \frac{P' J_0(\xi' r') J_1(\xi')}{t_1' \xi' + t_2' \xi' + t_3'} \left\{ (\eta_2 \xi' + \delta') \left[ \left( F'(\delta' \xi' + \gamma) - \frac{C'(\delta' \xi' + \gamma) k_1}{\nu_1} \right) e^{-\xi' \xi_1} \right] \right.
- (\eta_1 \xi' + \delta') \left[ \left( F'(\delta' \xi' + \gamma) - \frac{C'(\delta' \xi' + \gamma) k_2}{\nu_2} \right) e^{-\xi' \xi_2} \right] \right\} e^{-\xi' \xi_3'} d\xi',
\]
\[
\left. \right. (8.6)
\]
\[
\frac{u_z}{a} = g^{-\frac{1}{2}} \int_0^\infty \xi' \left[ k_1 \left( -\xi' A'_1 e^{-\xi' \xi_1} + \frac{\xi' B'_1}{\sqrt{\nu_1}} e^{\xi' \xi_1} \right) + k_2 \left( -\xi' A'_2 e^{-\xi' \xi_2} + \frac{\xi' B'_2}{\sqrt{\nu_2}} e^{\xi' \xi_2} \right) \right] J_0(\xi' r') d\xi',
\]
\[
(8.7)
\]

For the purposes of obtaining numerical values of the stress and displacement particular non-dimensional material constants for a transversely isotropic material which satisfy (7.1) are taken to be \( A' = 16.2, C' = 18.1, F' = 5.4, N' = A/3, L' = F' \) or equivalently \( E'_1 = 13.6, E'_2 = 15.4, G'_1 = 5.4, \nu_1 = 0.26, \) and \( \nu_2 = 0.25 \). Also the applied normal force is taken to be \( P' = 1/\pi \). Using these values, (8.5)–(8.7) were employed to obtain numerical values of the non-dimensional stress and displacement for the problems considered in sections 4 and 5.

To obtain the numerical values, the Newton–Cotes integration formula (see (19), formula 25.4.19) with a stepsize of \( \Delta = 0.05 \) was employed to evaluate the infinite integrals. The infinite interval of integration was approximated by a finite integral which was extended until the integral converged to eight decimal places. The interval required to achieve convergence varied for the different values of \( \delta', \gamma, \xi' \) and \( h' \) in Figs 1 to 8.

For the half-space problem of section 4, numerical values of the surface displacement \( u_z/a \) are displayed for several values of \( \delta' \) and \( \gamma \) in Fig. 1. The results illustrate that outside of the loaded

Fig. 1 Surface displacement \( u_z/a \) for an inhomogeneous half-space I
region, there is a narrowing of the interval in which there is substantial normal displacement as the inhomogeneity increases. This feature of the effect of an inhomogeneity has been commented on by Ward et al. (20) in connection with their field experiments and analysis concerning the suitability of a particular site for a large proton accelerator. Also using the values \( \delta' = 1 \) and \( \gamma = 1 \), numerical values of the displacement \( u_z/a \) and stress \( \sigma_{zz}/S \) at various points beneath the surface are shown in Figs 4 and 5. In Fig. 2, values of the surface displacement \( u_z/a \) are given to illustrate the movement of the surface displacement towards the values given by (4.35) for the case when \( \delta' \gamma = m = 1 \) and \( \gamma \) tends to zero. Specifically, the value of \( u_z/a \) in Fig. 2 for \( 0 < r' < 1 \), \( \delta' = 10 \) and \( \gamma = 0.1 \) may be compared with the value 0.017586 calculated from (4.35) with \( P' = 1/\pi \), \( m = 1 \) and \( C' = 18.1 \).
Figure 3 gives the surface displacement when \( \delta' = 1 \) and \( \gamma = 0.1 \), 0.02 and 0.01. This illustrates both the increase in the displacement as \( \gamma \) tends to zero with the value of \( \delta' \) held constant and also demonstrates the movement towards the values given by (4.35) as the ratio \( \gamma / \delta' \) decreases. The relevant values are \( \frac{u_z}{a} = 0.175862 \) when \( \gamma = 0.1 \), 0.879309 when \( \gamma = 0.02 \) and 1.758618 when \( \gamma = 0.01 \).

For the layer problem with \( p(r) = -P \) for \( 0 \leq r < a \) and \( p(r) = 0 \) for \( r > a \) (so that \( \bar{p}(\xi) \) is given by (4.20)), values of the surface displacement \( \frac{u_z}{a} \) are given in Figs 6 and 7 for a layer adhering to a rigid foundation with \( \delta' \gamma = m = 1 \) and various values of \( \delta' \) and \( h' \). Comparison
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Fig. 6 Surface displacement $u_z/a$ for an inhomogeneous layer with thickness $h' = 1$

Fig. 7 Surface displacement $u_z/a$ for inhomogeneous layers with $\delta' = 5$ and $\gamma = 0.2$

of Figs 6 and 2 shows that the surface displacement for $\delta' = 2$ and $\gamma = 0.5$ and also for $\delta' = 1$ and $\gamma = 1$ for the layer is less than the corresponding surface displacement for a half-space. For $\delta' = 10$ and $\gamma = 0.1$, the surface displacement for the layer and the half-space are both close to the displacement given by (4.35). Also Fig. 8 illustrates the effect of reducing the layer thickness on the surface displacement $u_z/a$.

For the isotropic case, the values of the constants in (6.14) are chosen to be $\lambda' = 5.4$ and $\epsilon' = 0.001$. Numerical values of the surface normal displacement for an inhomogeneous isotropic half-space with the above elastic moduli were calculated using (8.5). The results are shown in Fig. 8. They may be compared with the corresponding results for an anisotropic half-space displayed in
Fig. 1. The results in Fig. 1 are for an anisotropic material whose constants $c'_{ijkl}$ are all greater than or equal to the corresponding isotropic constants as given by (6.14) with $\lambda' = 5.4$. In particular, for the anisotropic material $c'_{3333} = C' = 18.1$, while from (6.14) in the isotropic case $c'_{3333} = C' = 3\lambda' = 16.2$. Thus, the isotropic material could be expected to give a larger surface displacement $u_z/a$ than the corresponding problem for the anisotropic material. The results in Figs 1 and 8 confirm that this is the case for the problem of section 4.

Also the displacement in Fig. 8 is consistent with the closed form formulas given in Johnson (21) for the displacement at the centre and the boundary of the circular loaded region on a homogeneous isotropic half-space. Johnson gives the formulas $u_z/a = 2(1 - \nu^2)P'/E'$ and $u_z/a = 4(1 - \nu^2)P'/\pi E'$ for the surface displacement at the centre $r' = 0$ and boundary $r' = 1$, respectively, of the loaded region. Specifically, with $\nu = 0.25$, $P' = 1/\pi$ and $E' = 5\lambda'/2 = 13.5$, the first of these formulas gives the value $u_z/a = 0.044$ and the second $u_z/a = 0.028$. These values are in accord with the values of $u_z/a$ for $r' = 0$ and $r' = 1$ shown on the graph for the homogeneous isotropic case when $\delta' = 0$.

9. Final remarks

Some axisymmetric problems have been considered for a restricted class of transversely isotropic half-spaces and layers in which the elastic moduli vary quadratically with the spatial variables. For the class of problems, considered formulas for the displacement are given in integral formulations that readily yield numerical values for particular problems and provide closed form formulas in limiting cases when the elastic moduli become small near the boundary of the half-space. The analysis can be used to consider the corresponding problems for isotropic materials as a particular case of the general anisotropic analysis. The analytical and numerical results obtained indicate the effect of material anisotropy and inhomogeneity on the deformation of particular transversely isotropic elastic half-spaces.
On the surface of the half-space, the axisymmetric problems considered exhibit a number of characteristics that are similar to the features observed by Gibson (1) and Gibson et al. (2) for the corresponding problems for an inhomogeneous incompressible isotropic half-space and layer in which the elastic moduli vary linearly with a spatial variable.

Finally, the analysis the paper is applicable to a restricted class of transversely isotropic materials. For such materials, the analysis provides exact solutions to the problems stated in section 2. In addition, the restricted class may be used to closely model an important class of transversely isotropic materials relevant in the study of intact rocks. In such cases, the analysis of this paper could be expected to give a useful solution to the problems of section 2.

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