Convolution and Correlation Theorem for Linear Canonical Transform and Properties

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Abstract

In this paper we introduce the convolution theorem for the linear canonical transform (LCT). Based on the properties of convolution theorem for the Fourier transform we explicitly show some important properties of the relationship between the LCT and convolution. We provide an alternative form of the correlation theorem for the LCT. We finally establish a theorem which describes the relationship between the LCT and the correlation.

Keywords: linear canonical transform, convolution, correlation

1 Introduction

The linear canonical transform (LCT) \cite{1, 6, 7, 8} is a linear integral transform with three free parameters which has found many applications in several areas, including signal processing and optics. It can be regarded as generalization of many transforms such as the Fourier transform, Laplace transform, the fractional Fourier transform, the Fresnel transform and the other transforms. Many properties of this transform are already known, including shift, modulation, and uncertainty principle \cite{11, 13, 14}. The most fundamental and important properties of the LCT are convolution and correlation. They are mathematical operations with several applications in pure and applied mathematics such as numerical analysis, numerical linear algebra and the design and implementation of finite impulse response filters in signal processing. However, there are still some problems concerning the convolution and correlation for the LCT. Although \cite{7, 10} discussed the formulas for convolution and correlation for the LCT, there is no widely accepted closed-form expression.

In this paper, we consider the alternative form of the correlation theorem for the LCT and establish properties of relationship between convolution and correlation theorems for the LCT, which have not been established in \cite{2, 3, 10}. We find that some properties of convolution theorem for the FT are still valid for the LCT, with some modification.

The paper is organized as follows: Section 2 briefly reviews the preliminaries about the convolution and correlation for the FT and their properties. The basic properties of the LCT are presented in Section 3. The definition of the convolution of the LCT is introduced and...
its important properties are also investigated in Section 4. The construction of correlation theorem of the LCT is presented in Section 5.

2 Some Properties of Convolution dan Correlation for FT

In this paper, we will deal with complex-valued function. Let us review the definition of the FT and collect some properties of relationship among convolution, correlation and the FT. For a complex-valued function \( f \) defined on the real line \( \mathbb{R} \), the complex conjugate function \( \overline{f} \) of \( f \) is given by

\[
\overline{f}(x) = \overline{f(x)}, \quad x \in \mathbb{R}.
\]

**Definition 2.1 (Fourier Transform).** Let \( f \in L^1(\mathbb{R}) \). The Fourier transform of \( f \) is defined by

\[
\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx.
\]

In the following, we introduce the fundamental operators and their basic properties, which will be used in the next section. Let \( f \in L^1(\mathbb{R}) \) and \( \omega_0 \in \mathbb{R} \). Denoted by \( \gamma_k f(x) = f(x - k) \), the shift by \( k \in \mathbb{R} \) of \( f \) and by \( M_{\omega_0} f(x) = e^{i\omega_0 x} f(x) \), the modulation by \( \omega_0 \) of \( f \).

a. Linearity:

\[
\mathcal{F}(\alpha f + \beta g)(\omega) = \alpha \mathcal{F}(f)(\omega) + \beta \mathcal{F}(g)(\omega), \quad \alpha, \beta \in \mathbb{C}.
\]

b. Shift:

\[
\mathcal{F}(\gamma_k f)(\omega) = e^{-i\omega k} \mathcal{F}(f)(\omega).
\]

c. Modulation:

\[
\mathcal{F}(M_{\omega_0} f)(\omega) = \mathcal{F}(f)(\omega - \omega_0).
\]

d. Time-frequency shift:

\[
\mathcal{F}(M_{\omega_0} \gamma_k f)(\omega) = e^{-i\omega_0 k} \mathcal{F}(f)(\omega - \omega_0).
\]

**Theorem 2.1 (Inversion Formula).** Suppose that \( f \in L^1(\mathbb{R}) \) and \( \mathcal{F}(f) \in L^1(\mathbb{R}) \), the inverse FT of \( f \) is given by

\[
\mathcal{F}^{-1}[\mathcal{F}(f)](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega.
\]

Two closely-related operations that are very important in the theory of linear time-invariant (LTI) systems are convolution and correlation. Let us introduce the definition of the convolution of two complex functions.

**Definition 2.2 (Convolution).** For \( f, g \in L^1(\mathbb{R}) \) the convolution \( f \ast g \) of \( f \) and \( g \) is defined by

\[
(f \ast g)(x) = \int_{-\infty}^{\infty} f(t)g(x - t) \, dt.
\]

**Theorem 2.2.** Suppose that \( f, g \in L^1(\mathbb{R}) \). Then the FT of convolution of two complex functions is given by

\[
\mathcal{F}(f \ast g)(\omega) = \sqrt{2\pi} \mathcal{F}(f) \mathcal{F}(g)(\omega).
\]
Let us collect useful properties of relationship between the convolution theorem and the FT.

**Theorem 2.3.** Suppose that \( f, g \in L^1(\mathbb{R}) \). Then the FT of \( \tau_k f \ast g \) and \( f \ast \tau_k g \) are the same and given by

\[
\mathcal{F}\{\tau_k f \ast g\}(\omega) = \mathcal{F}\{f \ast \tau_k g\}(\omega) = \sqrt{2\pi}e^{-i\omega k}\mathcal{F}\{f\}(\omega)\mathcal{F}\{g\}(\omega).
\]

(10)

We also have

\[
\mathcal{F}\{f \ast M_{\omega_0} g\}(\omega) = \sqrt{2\pi}\mathcal{F}\{f\}(\omega)\mathcal{F}\{g\}(\omega - \omega_0)
\]

(11)

\[
\mathcal{F}\{M_{\omega_0} f \ast g\}(\omega) = \sqrt{2\pi}\mathcal{F}\{f\}(\omega - \omega_0)\mathcal{F}\{g\}(\omega).
\]

(12)

Next, let us examine how the FT behaves under correlations. Firstly, we give the definition of the correlation of two complex functions.

**Definition 2.3 (Correlation).** Suppose that \( f, g \in L^1(\mathbb{R}) \), the correlation of two complex functions is defined by

\[
(f \ast g)(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)g(t + x) dt.
\]

(14)

**Theorem 2.4.** Suppose that \( f, g \in L^1(\mathbb{R}) \). Then the FT of correlation of \( f \) and \( g \) is given by

\[
(f \ast g)(\omega) = \sqrt{2\pi}\mathcal{F}\{f\}(-\omega)\mathcal{F}\{g\}(\omega).
\]

(15)

Or, equivalently,

\[
(f \ast g)(\omega) = \sqrt{2\pi}\mathcal{F}\{f\}(-\omega)\mathcal{F}\{g\}(\omega).
\]

(16)

Some useful properties of the correlation theorem for the FT are summarized in the following theorem.

**Theorem 2.5.** Let \( f, g \in L^1(\mathbb{R}) \), then the FT of \( \tau_k f \ast g \) and \( f \ast \tau_k g \) is given by

\[
\mathcal{F}\{\tau_k f \ast g\}(\omega) = \sqrt{2\pi}e^{i\omega k}\mathcal{F}\{f\}(\omega)\mathcal{F}\{g\}(\omega)
\]

(17)

\[
\mathcal{F}\{f \ast \tau_k g\}(\omega) = \sqrt{2\pi}ie^{-i\omega k}\mathcal{F}\{f\}(\omega)\mathcal{F}\{g\}(\omega).
\]

(18)

We also have

\[
\mathcal{F}\{f \ast M_{\omega_0} g\}(\omega) = \sqrt{2\pi}\mathcal{F}\{f\}(\omega - \omega_0)\mathcal{F}\{g\}(\omega)
\]

(19)

\[
\mathcal{F}\{M_{\omega_0} f \ast g\}(\omega) = \sqrt{2\pi}\mathcal{F}\{f\}(\omega)\mathcal{F}\{g\}(\omega - \omega_0).
\]

(20)

As the proofs of all theorems mentioned above are straightforward, we omit the proofs.

3 Linear Canonical Transform

Linear canonical transform (LCT) is a good tool to analyze signals in the frequency domains. It was firstly introduced by Moshinski and Collins [3, 5]. This section introduces the definition of the LCT and some its useful properties. We use the following matrix notation: \( A = (a, b; c, d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2 \times 2} \). Denoted by \( SL(2, \mathbb{R}) \), the special linear group of degree 2 over \( \mathbb{R} \), that is, the set \( 2 \times 2 \) matrices \( A = (a, b; c, d) \) with \( \det(A) = ad - bc = 1 \).
Definition 3.1 (LCT). Let $A = (a, b; c, d) \in SL(2, \mathbb{R})$. The LCT of a signal $f \in L^1(\mathbb{R})$ is defined by

$$F_A(\omega) = L_A\{f\}(\omega) = \begin{cases} \int_{-\infty}^{\infty} f(x)K_A(\omega, x) \, dx, & b \neq 0, \\ \sqrt{\det(\frac{1}{i})} \omega^2 (fdu), & b = 0, \end{cases}$$

(21)

where

$$K_A(\omega, x) = \frac{1}{\sqrt{2\pi ib}} e^{i\left(\frac{x}{2b^2} - \frac{x}{2} \omega^2\right)}.$$

(22)

From the definition of the LCT, we can see easily that, when $b = 0$, the LCT of a signal is essentially a chirp multiplication. Therefore, in this paper, we always assume $b \neq 0$.

Remark 3.1. It should be remembered that the entries of the matrix parameter $A = (a, b, c, d)$ are real or complex numbers and for simplicity we assume that they are real numbers.

As a special case, when $A = (a, b; c, d) = (0, 1; -1, 0)$, the LCT definition (21) reduces to the FT definition (2). The inverse transform of the LCT is given by

$$f(x) = \sqrt{\frac{i}{2\pi b}} \int_{-\infty}^{\infty} F_A\{f\}(\omega) K_A^{-1}(\omega, x) \, d\omega.$$

Or, equivalently,

$$f(x) = \sqrt{\frac{i}{2\pi b}} \int_{-\infty}^{\infty} F_A\{f\}(\omega) e^{i\left(\frac{1}{2} \omega^2 + \frac{x}{2} \omega - \frac{x^2}{2}\right)} \, d\omega,$$

(24)

where $A^{-1} = (d, -b; -c, a)$. It is not difficult to see that the relationship between LCT and the FT is given by

$$F_A(\omega) = L_A\{f\}(\omega) = \frac{1}{\sqrt{4b}} e^{i\frac{a}{b} \omega^2} \mathcal{F}\{e^{\frac{b}{a} x^2} f(x)\}\left(\frac{\omega}{b}\right).$$

(25)

Example 3.1. Let us compute the LCT of the Gaussian function $f(x) = e^{-kx^2}$ with $k > 0$. From the definition of the LCT (21), we easily get

$$L_A\{f\}(\omega) = \frac{1}{\sqrt{2\pi bi}} \int_{-\infty}^{\infty} f(x) e^{i\left(\frac{1}{2b^2} - \frac{x}{2} \omega^2\right)} \, dx$$

$$= \frac{1}{\sqrt{2\pi bi}} \int_{-\infty}^{\infty} e^{-kx^2} e^{i\frac{1}{2b^2} x^2} e^{-i\frac{x}{2} \omega^2} \, dx$$

$$= \frac{1}{\sqrt{2\pi bi}} e^{i\frac{1}{2b^2} \omega^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2b} (2kb - ia)x^2} e^{-i\frac{x}{2} \omega} \, dx.$$

(26)

Using the FT of the Gaussian function

$$\mathcal{F}\{f\}(\omega) = \int_{-\infty}^{\infty} e^{-kx^2} e^{-i\omega x} \, dx = \sqrt{\frac{\pi}{k}} e^{-\frac{k^2}{4k}}, \quad k > 0.$$

(27)

We immediately obtain

$$L_A\{f\}(\omega) = \frac{1}{\sqrt{2\pi bi}} e^{i\frac{1}{2b^2} \omega^2} \sqrt{\frac{2\pi b}{(2kb - ia)}} e^{-\frac{k^2}{2b(2kb - ia)}}.$$

-2512-
Let us list the important properties of the LCT and give their proofs in detail. We see that the results are generalizations of basic properties of the FT. Let us begin with the shift property of LCT.

**Theorem 3.1 (Shift property).** Let \( f \in L^1(\mathbb{R}). \) The LCT of a shift by \( k \in \mathbb{R} \) is given by

\[
L_A \{ \tau_k f \}(\omega) = L_A \{ f(x - k) \}(\omega) = e^{-\frac{iask^2}{2} + ick\omega} F_A(\omega - ak). \tag{28}
\]

**Proof.** We have from the definition of the LCT (21)

\[
L_A \{ \tau_k f \}(\omega) = \int_{-\infty}^{\infty} f(x - k) \frac{1}{\sqrt{2\pi i}b} e^{i\frac{1}{2} \left( \frac{x}{b} \right)^2 - \frac{i}{2} \left( \frac{x}{b} \right) \omega + \frac{i}{2} \omega^2} dx. \tag{29}
\]

By making the change of a variable \( x - k = m, \) we easily obtain

\[
L_A \{ \tau_k f \}(\omega) = \int_{-\infty}^{\infty} f(m) \frac{1}{\sqrt{2\pi i}b} e^{i\frac{1}{2} \left( \frac{m+k}{b} \right)^2 - \frac{i}{2} \left( \frac{m+k}{b} \right) \omega + \frac{i}{2} \omega^2} dm.
\]

Therefore, we further get

\[
L_A \{ \tau_k f \}(\omega) = \int_{-\infty}^{\infty} f(m) e^{i\frac{1}{2} \left( \frac{m}{b} \right)^2} e^{i\left( \frac{mk}{b} \right) \omega} e^{i(\frac{1}{2} k^2)} e^{-\frac{i}{2} \left( \frac{m^2}{b} \right) (\omega - ka) + \frac{i}{2} (\omega - ka)^2} dm.
\]

Therefore, we further get

\[
L_A \{ \tau_k f \}(\omega) = \int_{-\infty}^{\infty} f(m) e^{i\frac{1}{2} \left( \frac{m}{b} \right)^2} e^{i\left( \frac{mk}{b} \right) \omega} e^{i(\frac{1}{2} k^2)} e^{-\frac{i}{2} \left( \frac{m^2}{b} \right) (\omega - ka) + \frac{i}{2} (\omega - ka)^2} dm.
\]
Applying the definition of the LeT (21), the above expression can be rewritten in the form

\[ L_A\{\tau_k f\}(\omega) = e^{\frac{i}{2} \frac{\omega}{\hbar} (2(\omega - ka)k + (ka)^2)} e^{\frac{i}{2} \frac{\omega}{\hbar} k^2} e^{\frac{i}{2} \frac{\omega}{\hbar} k^2} F_A(\omega - ka). \]  

(30)

We notice that

\[
e^{\frac{i}{2} \frac{\omega}{\hbar} (2(\omega - ka)k + (ka)^2)} e^{\frac{i}{2} \frac{\omega}{\hbar} k^2} e^{\frac{i}{2} \frac{\omega}{\hbar} k^2} e^{\frac{i}{2} \frac{\omega}{\hbar} k^2} = e^{\frac{i}{2} \frac{\omega}{\hbar} (2\omega ka - 2(ka)^2 + (ka)^2)} e^{\frac{i}{2} \frac{\omega}{\hbar} k^2} e^{\frac{i}{2} \frac{\omega}{\hbar} k^2} = e^{\frac{i}{2} \frac{\omega}{\hbar} ka} e^{-i \frac{\omega}{\hbar} (ka)^2} e^{\frac{i}{2} \frac{\omega}{\hbar} k^2} e^{\frac{i}{2} \frac{\omega}{\hbar} k^2} = e^{ik\omega} e^{-\frac{i}{2} \hbar^2 c}.
\]

We finally arrive at

\[ L_A\{\tau_k f\}(\omega) = e^{ik\omega} e^{-\frac{i}{2} \hbar^2 c} F_A(\omega - ka), \]  

(32)

which completes the proof.

Theorem 3.2 (Modulation property). Let \( f \in L^1(\mathbb{R}) \). The LeT of a modulation by \( \omega_0 \in \mathbb{R} \) is given by

\[ L_A\{M_{\omega_0} f\}(\omega) = L_A\{e^{i\omega_0 x} f(x)\}(\omega) = e^{-\frac{ib\omega_0^2}{2} + i\omega_0 \omega} F_A(\omega - b\omega_0). \]  

(33)

\[ L_A\{M_{\omega_0} f\}(\omega) = \int_{-\infty}^{\infty} e^{i\omega_0 x} f(x) e^{\frac{1}{\sqrt{2\pi \hbar}} (\frac{x^2}{2} - \frac{x\omega}{\hbar} + \frac{\omega^2}{2})} dx \]

\[ = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} f(x) e^{\frac{1}{2} (\frac{x^2}{2} + \frac{x\omega}{\hbar} + \frac{\omega^2}{2})} \]  

(34)

\[ = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} f(x) e^{\frac{1}{2} (\frac{x^2}{2} - (2x(\omega - \omega_0)) + \frac{\omega^2}{2})} dx = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} f(x) e^{\frac{1}{2} (\frac{x^2}{2} - \frac{x}{2}(\omega - \omega_0)) + \frac{\omega^2}{2}} dx. \]

The right side of (35) can be rewritten in the following form

\[ L_A\{M_{\omega_0} f\}(\omega) = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} f(x) e^{\frac{1}{2} (\frac{x^2}{2} - \frac{x}{2}(\omega - \omega_0) + \frac{\omega^2}{2})} \]

\[ = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} f(x) e^{\frac{1}{2} (\frac{x^2}{2} - \frac{x}{2}(\omega - \omega_0) + \frac{\omega^2}{2})} \]  

(35)

\[ = e^{i(\omega - \omega_0) \omega_0 + \frac{ib\omega_0^2}{2}} F_A(\omega - \omega_0 b) = e^{i\omega_0 \omega_0 + \frac{ib\omega_0^2}{2}} F_A(\omega - \omega_0 b) = e^{i\omega_0 \omega_0 + \frac{ib\omega_0^2}{2}} F_A(\omega - \omega_0 b), \]

(36)

which is desired result.
Theorem 3.3 (Time-frequency shift). Let \( f \in L^1(\mathbb{R}) \). Then we get

\[
L_A(\mathcal{M}_{\omega_0 \tau_k} f)(\omega) = L_A(e^{i\omega_0 f(x - k)})(\omega) = e^{-i(ak^2 + b\omega b_0)/2 + i((ck + d\omega_0)\omega - ibck\omega_0)} L_A(f)(\omega - ak - b\omega_0).
\]

(37)

Proof. Since \( b \neq 0 \), by applying the definition of the LCT (21), we get

\[
L_A(\mathcal{M}_{\omega_0 \tau_k} f)(\omega) = \int_{-\infty}^{\infty} e^{ix\omega_0} f(x - k) \frac{1}{\sqrt{2\pi b}} e^{i\frac{1}{2}((\frac{\omega}{2}x^2 - \frac{\omega}{2}z\omega + \frac{\omega^2}{2} + \omega^2)dx}
\]

\[
= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(x - k) e^{i\frac{1}{2}((\frac{\omega}{2}x^2 - \frac{\omega}{2}z\omega + \frac{\omega^2}{2} + \omega^2)dx}
\]

\[
= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(x - k) e^{i\frac{1}{2}((\frac{\omega}{2}x^2 - \frac{\omega}{2}z\omega + \frac{\omega^2}{2} + \omega^2)dx}
\]

Letting \( \omega - \omega_0 b = m \), we get

\[
L_A(\mathcal{M}_{\omega_0 \tau_k} f)(\omega) = \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(x - k) e^{i\frac{1}{2}((\frac{\omega}{2}x^2 - \frac{\omega}{2}z\omega + \frac{\omega^2}{2} + \omega^2)dx}
\]

\[
= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(x - k) e^{i\frac{1}{2}((\frac{\omega}{2}x^2 - \frac{\omega}{2}z\omega + \frac{\omega^2}{2} + \omega^2)dx}
\]

Setting \( x - k = n \), the right side of the above expression can be rewritten in the form

\[
L_A(\mathcal{M}_{\omega_0 \tau_k} f)(\omega)
\]

\[
= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(n) e^{i\frac{1}{2}((\frac{\omega}{2}(n+k)^2 - \frac{\omega}{2}(n+k)m + \frac{\omega^2}{2}m^2) - \frac{\omega}{2}n^2)} e^{i\frac{1}{2}(\omega m + \omega b_0)dn}
\]

\[
= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(n) e^{i\frac{1}{2}((\frac{\omega}{2}(n^2 + 2nk + k^2) - \frac{\omega}{2}n^2 - \frac{\omega}{2}km + \frac{\omega^2}{2}m^2) - \frac{\omega}{2}n^2)} e^{i\frac{1}{2}(\omega m + \omega b_0)dn}
\]

Applying the definition of the LCT (21) yields

\[
L_A(\mathcal{M}_{\omega_0 \tau_k} f)(\omega)
\]

\[
= \left( e^{i\frac{1}{2}(\frac{\omega}{2}(2mak - (ak)^2))} e^{i\frac{1}{2}(\frac{\omega}{2}km)} e^{-i(\frac{1}{2})} \frac{e^{i\omega_0 (\omega - ak)}}{e^{-i\omega_0 (\omega - ak)}} \right) L_A(f)(m - ak)
\]

\[
= \left( e^{ikm(\frac{\omega}{\omega_0} - \frac{1}{2})} e^{-i\frac{1}{2}ak^2(\frac{\omega}{\omega_0} - \frac{1}{2})} \right) e^{i\frac{1}{2}(\omega m + \omega b_0)} L_A(f)(m - ak)
\]
\[ m = \omega - \omega_0 b, \]
Since \( m = \omega - \omega_0 b \), then we have

\[ \begin{align*}
L_A\{M_{\omega_0}k_f\}(\omega) &= e^{-i(\frac{1}{2})ak^2c}e^{-i(\frac{1}{2})bd\omega_0}L_A\{f\}(\omega - \omega_0 b - ak) \\
&= e^{-i(\frac{1}{2})ak^2c}e^{-i(\frac{1}{2})bd\omega_0}e^{i(\omega - \omega_0 b)(k_\omega + \omega_0 d)}L_A\{f\}(\omega - \omega_0 b - ak) \\
&= e^{-i(\frac{1}{2})ak^2c}e^{-i(\frac{1}{2})bd\omega_0}e^{i\omega k_\omega + \omega_0 d - \omega_0 b e^{-i\omega_0 d}}L_A\{f\}(\omega - \omega_0 b - ak) \\
&= e^{-i(\frac{1}{2})ak^2c}e^{-i(\frac{1}{2})bd\omega_0}e^{e^{i(\omega k_\omega + \omega_0 d) - \omega_0 b e^{-i\omega_0 d}}}L_A\{f\}(\omega - \omega_0 b - ak) \\
&= e^{-i(\frac{1}{2})ak^2c}e^{-i(\frac{1}{2})bd\omega_0}e^{e^{i(\omega k_\omega + \omega_0 d) - \omega_0 b e^{-i\omega_0 d}}}L_A\{f\}(\omega - \omega_0 b - ak),
\end{align*} \]
which completes the proof. \( \square \)

4 Convolution Theorem for LCT and Properties

In this section we investigate how the LCT behaves under convolutions. For this purpose, we define convolution for the LCT (see [1,2]).

**Definition 4.1.** For any two functions \( f, g \in L^1(\mathbb{R}) \), we define the convolution operator of the LCT as

\[ (f \ast g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)W(t,x) \, dt, \]
(38)

where \( W(t,x) = e^{it(t-x)/a/b} \). The LCT of the convolution of two complex functions is given by

\[ L_A\{f \ast g\} = \sqrt{2\pi b}ie^{-i(k_\omega^2)}L_A\{f\}(\omega)L_A\{g\}(\omega). \]
(39)

Some properties of the convolution for the LCT are collected by the following theorems.

**Theorem 4.1** (Shift Convolution). Let \( f, g \in L^1(\mathbb{R}) \). The LCT of \( \tau_k f \ast g \) and \( f \ast \tau_k g \) is the same and is given by

\[ L_A\{\tau_k f \ast g\}(\omega) = L_A\{f \ast \tau_k g\}(\omega) = \sqrt{2\pi b}ie^{-i(k_\omega^2)}e^{-iak^2/2} + i\omega d L_A\{f\}(\omega - ak)L_A\{g\}(\omega). \]
(40)

**Proof.** Direct computations show that

\[ L_A\{\tau_k f \ast g\}(\omega)\sqrt{\frac{1}{2\pi bi}}e^{i(k_\omega^2)} \]

\[ = \sqrt{\frac{1}{2\pi bi}} \int_{-\infty}^{\infty} f(t-k)e^{i\omega(t-k)}e^{-i(k_\omega^2)} \, dt \sqrt{\frac{1}{2\pi bi}} \int_{-\infty}^{\infty} g(\tau)e^{i\omega(\tau+k)}e^{-i(k_\omega^2)} \, d\tau. \]

It implies that

\[ L_A\{\tau_k f \ast g\}(\omega)\sqrt{\frac{1}{2\pi bi}}e^{i(k_\omega^2)} = L_A\{f(x-k)\}(\omega)L_A\{g\}(\omega). \]
(41)
Or, equivalently,
\[ L_A \{ \tau_k f * g \}(\omega) = \sqrt{2\pi} b e^{i(-\frac{d^2}{2})(\frac{d^2}{2})} L_A \{ \tau_k f \}(\omega) L_A \{ g \}(\omega). \] (42)

Since \( L_A \{ \tau_k f * g \}(\omega) = e^{-\frac{d^2}{2} + ic\omega} L_A \{ f \}(\omega - ak) \), we have
\[ L_A \{ \tau_k f * g \}(\omega) = \sqrt{2\pi} b e^{i(-\frac{d^2}{2})} e^{-\frac{d^2}{2} + ic\omega} L_A \{ f \}(\omega - ak) L_A \{ g \}(\omega), \] (43)
which proves the theorem.

**Theorem 4.2** (Modulation Convolution). For arbitrary \( f, g \in L^1(\mathbb{R}) \), we have

\[ L_A \{ M_{\omega_0} f * g \}(\omega) = \sqrt{2\pi} b e^{-i\frac{d}{2}(\omega - \omega_0)^2} L_A \{ f \}(\omega - \omega_0) L_A \{ g \}(\omega), \] (44)
\[ L_A \{ f * M_{\omega_0} g \}(\omega) = \sqrt{2\pi} b e^{-i\frac{d}{2}(\omega - \omega_0)^2} L_A \{ f \}(\omega) L_A \{ g \}(\omega - \omega_0). \] (45)

**Proof.** For the proof of (44), we have
\[ (M_{\omega_0} f * g)(\omega) = \int_{-\infty}^{\infty} e^{i\omega_0 t} f(t) g(x - t) W(t, x) \ dt. \] (46)
Taking the LCT to both sides of (46) we obtain
\[ L_A \{ M_{\omega_0} f * g \}(\omega) = \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega_0 t} f(t) g(x - t) W(t, x) \ dt \ dx \] (47)
\[ = \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega_0 t} f(t) g(x - t) e^{-i\frac{d}{2}(x - t)^2} \ dx \ dt. \]
Making a change of variable \( x - t = v \) in above expression, we have
\[ L_A \{ M_{\omega_0} f * g \}(\omega) = \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega_0 t} f(t) g(v) e^{-i\frac{d}{2} v^2} e^{-i\frac{d}{2}(v + t)^2} \ dv \ dt. \] (48)
Multiplying both sides of (47) by \( e^{i\frac{d}{2}(x - t)^2} \) and applying the definition of the LCT we, immediately obtain
\[ L_A \{ M_{\omega_0} f * g \}(\omega) = \sqrt{2\pi} b e^{-i\frac{d}{2}(\omega - \omega_0)^2} L_A \{ f \}(\omega - \omega_0) L_A \{ g \}(\omega), \] (49)
which completes the proof.

**Theorem 4.3** (Time-frequency shift convolution). Suppose that \( f, g \in L^1(\mathbb{R}) \). The LCT of \( M_{\omega_0} \tau_k f * g \) and \( M_{\omega_0} f * \tau_k g \) is given by
\[ L_A \{ M_{\omega_0} \tau_k f * g \}(\omega) = \sqrt{2\pi} b e^{-i\frac{d}{2}(\omega - \omega_0)^2} e^{-i\frac{d}{2}k^2 + ic\omega} L_A \{ f \}(\omega - \omega_0 - ak) L_A \{ g \}(\omega), \] (50)
\[ L_A \{ f * M_{\omega_0} \tau_k g \}(\omega) = \sqrt{2\pi} b e^{-i\frac{d}{2}(\omega - \omega_0)^2} e^{-i\frac{d}{2}k^2 + ic\omega} L_A \{ f \}(\omega) L_A \{ g \}(\omega - \omega_0 - ak). \] (51)
Proof. We only prove (49), since the other is similar. Applying the LCT definition and LCT convolution, we have

\[
L_A\{M_{\omega_0} \tau_k f \ast g\}(w) = \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega_0 t} f(t-k)g(x-t) e^{-i\omega (x-t)^2/b} \, dt \, dx
\]

Putting \(x-t=v\), we obtain

\[
L_A\{M_{\omega_0} \tau_k f \ast g\}(\omega) = \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega_0 t} f(t-k)g(v) e^{-i\omega (v-t)^2/b} e^{i\omega v^2/b} \, dt \, dv
\]

Multiplying both sides of the above identity by \(e^{i\omega_0 (w-W_0)^2}\), we immediately get

\[
L_A\{M_{\omega_0} \tau_k f \ast g\}(w) = \frac{1}{\sqrt{2\pi b}} e^{-i\omega (w-W_0)^2} L_A\{f(t-k)\}(w-W_0) L_A\{g\}(w)
\]

which finishes the proof.

5 Alternative Formulation of Correlation for LCT

In this section we give an alternative form of the correlation of the LCT (compare to [9]). We then investigate the relationship between the FT and the correlation of two complex functions.

Definition 5.1. For any two functions \(f, g \in L^1(\mathbb{R})\), we define the correlation operator of the LCT as

\[
(f \circ g)(x) = \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) g(t+x) e^{i\omega(x^2/2b)} \, dt \, dx
\]

where \(W_{cr}(t,x) = e^{i\omega(x^2/2b)}\).

Theorem 5.1. For two complex-valued functions \(f, g \in L^1(\mathbb{R})\), we get the LCT of correlation of two functions

\[
L_A\{f \circ g\}(\omega) = \sqrt{2\pi b} e^{-i\omega(W_0/2b)} L_A\{f\}(-\omega) L_A\{g\}(\omega) - i \sqrt{2\pi b} e^{-i\omega(W_0/2b)} L_A\{f\}(-\omega) L_A\{g\}(\omega).
\]

When \(A = (a, b, c, d) = (0,1; -1,0)\) the above expression coincides with the correlation theorem of the FT (15), i.e.,

\[
L_A\{f \circ g\}(\omega) = \sqrt{2\pi b} L_A\{f\}(-\omega) L_A\{g\}(\omega) - i \sqrt{2\pi b} L_A\{f\}(-\omega) L_A\{g\}(\omega).
\]

Proof. From the definitions of the LCT (21) and the LCT correlation we obtain

\[
L_A\{f \circ g\}(\omega) = \sqrt{2\pi b} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) g(t+x) \, dt \, dx e^{i\omega(x^2/2b)}
\]

\[
= \sqrt{2\pi b} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f_0(t) - i f_1(t)) g(t+x) \, dt \, dx e^{i\omega(x^2/2b)}
\]

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Setting $x + t = v$, we get

\[ L_A\{f \circ g\}(\omega) = \sqrt{\frac{1}{2\pi i}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f_0(t) - if_1(t))g(t + x)e^{i\left(\frac{(a(x - t))^2}{2b}\right)}e^{-i\left(\frac{(a(x - t))^2}{2b}\right)}e^{i\left(\frac{dx^2}{25}\right)}e^{i\left(\frac{dx^2}{25}\right)}dtdv \]

\[ = \sqrt{\frac{1}{2\pi i}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_0(t)e^{i\left(\frac{(a(x - t))^2}{2b}\right)}e^{-i\left(\frac{(a(x - t))^2}{2b}\right)}g(v)e^{i\left(\frac{dx^2}{25}\right)}dtdv \]

\[ - i\sqrt{\frac{1}{2\pi i}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(t)e^{i\left(\frac{(a(x - t))^2}{2b}\right)}e^{-i\left(\frac{(a(x - t))^2}{2b}\right)}g(v)e^{i\left(\frac{dx^2}{25}\right)}dtdv \]

\[ = \sqrt{\frac{1}{2\pi i}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_0(t)e^{i\left(\frac{(a(x - t))^2}{2b}\right)}e^{-i\left(\frac{(a(x - t))^2}{2b}\right)}e^{i\left(\frac{dx^2}{25}\right)}g(v)e^{i\left(\frac{dx^2}{25}\right)}dtdv \]

\[ - i\sqrt{\frac{1}{2\pi i}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(t)e^{i\left(\frac{(a(x - t))^2}{2b}\right)}e^{-i\left(\frac{(a(x - t))^2}{2b}\right)}e^{i\left(\frac{dx^2}{25}\right)}g(v)e^{i\left(\frac{dx^2}{25}\right)}dtdv \]

\[ = \sqrt{\frac{1}{2\pi i}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_0(t)e^{i\left(\frac{a(x^2 + \nu t^2)}{2b}\right)}e^{-i\left(\frac{a(x^2 + \nu t^2)}{2b}\right)}e^{i\left(\frac{dx^2}{25}\right)}g(v)e^{i\left(\frac{dx^2}{25}\right)}dtdv \]

\[ - i\sqrt{\frac{1}{2\pi i}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(t)e^{i\left(\frac{a(x^2 + \nu t^2)}{2b}\right)}e^{-i\left(\frac{a(x^2 + \nu t^2)}{2b}\right)}e^{i\left(\frac{dx^2}{25}\right)}g(v)e^{i\left(\frac{dx^2}{25}\right)}dtdv \]

Applying the definition of the LCT yields

\[ L_A\{f \circ g\}(\omega) = L_A\{f_0\}(-\omega) \int_{-\infty}^{\infty} g(v)e^{i\left(\frac{a(x^2)}{2b}\right)}e^{-i\left(\frac{a(x^2)}{2b}\right)}dv \]

\[ - iL_A\{f_1\}(-\omega) \int_{-\infty}^{\infty} g(v)e^{i\left(\frac{a(x^2)}{25}\right)}e^{-i\left(\frac{a(x^2)}{25}\right)}dv. \quad (57) \]

Multiplying both sides of (57) by $e^{i\left(\frac{dx^2}{25}\right)}$, we easily obtain

\[ L_A\{f \circ g\}(\omega) \sqrt{\frac{1}{2\pi i}} e^{i\left(\frac{dx^2}{25}\right)} \]

\[ = \sqrt{\frac{1}{2\pi i}} e^{i\left(\frac{dx^2}{25}\right)} L_A\{f_0\}(-\omega) \int_{-\infty}^{\infty} g(v)e^{i\left(\frac{a(x^2)}{2b}\right)}e^{-i\left(\frac{a(x^2)}{2b}\right)}dv \]

\[ - i\sqrt{\frac{1}{2\pi i}} e^{i\left(\frac{dx^2}{25}\right)} L_A\{f_1\}(-\omega) \int_{-\infty}^{\infty} g(v)e^{i\left(\frac{a(x^2)}{2b}\right)}e^{-i\left(\frac{a(x^2)}{2b}\right)}dv \]

\[ = L_A\{f_0\}(-\omega) \sqrt{\frac{1}{2\pi i}} \int_{-\infty}^{\infty} g(v)e^{i\left(\frac{a(x^2)}{2b}\right)}e^{-i\left(\frac{a(x^2)}{2b}\right)}dv \]

\[ \int_{-\infty}^{\infty} g(v)e^{i\left(\frac{a(x^2)}{25}\right)}e^{-i\left(\frac{a(x^2)}{25}\right)}dv. \]
The definition of the LCT (21) gives

$$
-L_A(f_1)(-\omega)\sqrt{\frac{1}{2\pi bi}}\int_{-\infty}^{\infty} g(v)e^{i\left(\frac{v^2}{2b}\right)}e^{-i\left(\frac{wv}{b}\right)}dv.
$$

(58)

Or, equivalently,

$$
L_A(f \circ g)(\omega)\sqrt{\frac{1}{2\pi bi}}e^{i\left(\frac{\omega^2}{2b}\right)} = L_A(f_0)(-\omega)L_A(g)(\omega) - iL_A(f_1)(-\omega)L_A(g)(\omega).
$$

This ends the proof of the theorem. \qed

References


CONVOLUTION AND CORRELATION THEOREM FOR LINEAR CANONICAL TRANSFORM


