



CENTER OF THE SKEW POLYNOMIAL RING OVER TRINION MATRIX

**Amir Kamal Amir, Nur Erawaty, Mawardi Bahri and
Aidah Nabilah Anwar**

Department of Mathematics
Hasanuddin University
Indonesia
e-mail: amirkamalamir@yahoo.com

Abstract

Let R be a ring and σ be an endomorphism on R . Then the set of polynomials $\{f(t) = a_0 + a_1t + \dots + a_nt^n \mid a_i \in R\}$ forms a ring with multiplication rule $ta = \sigma(a)t$ for all $a \in R$. This multiplication rule is noncommutative. Therefore, the set of polynomials described is called skew polynomial ring. In this paper, we consider R to be the 2 by 2 trinion matrix and determine its center.

1. Definitions and Notations

Trinions are elements of a 3-dimensional associative algebra. The set of trinions can be represented as

$$\mathbb{T} = \{a_0 + a_1\mathbf{i} + a_2\mathbf{j} \mid a_0, a_1, a_2 \in \mathbb{R}\},$$

where

$$\mathbf{i}^2 = \mathbf{j}, \mathbf{j}^2 = -\mathbf{i}, \mathbf{ij} = \mathbf{ji} = -1.$$

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A trinion matrix is a matrix whose entries are trinions. A trinion matrix can be represented as

$$M = \begin{bmatrix} a_0 + a_1\mathbf{i} + a_2\mathbf{j}^{(11)} & \cdots & a_0 + a_1\mathbf{i} + a_2\mathbf{j}^{(1n)} \\ \vdots & \ddots & \vdots \\ a_0 + a_1\mathbf{i} + a_2\mathbf{j}^{(n1)} & \cdots & a_0 + a_1\mathbf{i} + a_2\mathbf{j}^{(nn)} \end{bmatrix}.$$

For more about trinions, the reader is referred to [1]. In this paper, it is necessary to recall the formal definition of a skew polynomial ring and the center of a ring. The following definitions can be found in [2-6].

Let R be a ring with identity element 1, σ be an endomorphism on R , and δ be a σ -derivation on the ring R . Then the skew polynomial ring over R with respect to the skew derivation (σ, δ) is a ring that consists of all polynomials over R in an indeterminate t denoted by,

$$R[t; \sigma, \delta] = \{f(t) = a_0 + a_1t + \cdots + a_nt^n \mid a_i \in R\},$$

satisfying the following equation, for all $a \in R$,

$$ta = \sigma(a)t + \delta(a).$$

The notation $R[t; \sigma]$ stands for the particular skew polynomial ring where $\delta = 0$. Moreover, the structure of the skew polynomial ring $R[t; \sigma]$ is different from $R[t; \sigma, \delta]$.

Let R be a ring. Then the center of R , denoted by $Z(R)$, is defined as:

$$Z(R) = \{r \in R \mid rx = xr, \forall x \in R\}.$$

2. The Main Results

In this section, we determine the center of skew polynomial ring of 2×2 trinion matrix.

Let $M = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \mid a, b \in \mathbb{T} \right\} \in M_{(\mathbb{T})}^{2 \times 2}$. Define $\sigma : M_{(\mathbb{T})}^{2 \times 2} \rightarrow M_{(\mathbb{T})}^{2 \times 2}$ by

$$\begin{aligned}\sigma \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} &= \sigma \begin{pmatrix} a_0 + a_1\mathbf{i} + a_2\mathbf{j} & 0 \\ b_0 + b_1\mathbf{i} + b_2\mathbf{j} & a_0 + a_1\mathbf{i} + a_2\mathbf{j} \end{pmatrix} \\ &= \begin{pmatrix} a_0 - a_2\mathbf{i} - a_1\mathbf{j} & 0 \\ b_0 - b_2\mathbf{i} - b_1\mathbf{j} & a_0 - a_2\mathbf{i} - a_1\mathbf{j} \end{pmatrix},\end{aligned}$$

where $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j}$ and $b = b_0 + b_1\mathbf{i} + b_2\mathbf{j}$.

Then it is easy to see that σ is an endomorphism.

Center of skew polynomial ring $M[t; \sigma]$ is described in the following theorem:

Theorem 2.1.

$$\begin{aligned}Z(M[t; \sigma]) &= \left\{ \sum_{n=0}^m \begin{pmatrix} a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) & \mathbf{0} \\ b_{2n} + b_{2n+2}(-\mathbf{i} + \mathbf{j}) & a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) \end{pmatrix} t^{2n} \right. \\ &\quad \left. + \begin{pmatrix} a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & \mathbf{0} \\ b_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) \end{pmatrix} t^{2n+1} \mid a_{2n}, b_{2n}, \right. \\ &\quad \left. a_{2n+1}, b_{2n+1}, a_{2n+2}, b_{2n+2} \in \mathbb{R} \right\}.\end{aligned}$$

Proof. The proof is divided into two parts, the first one shows that

$$\begin{aligned}&\left\{ \sum_{n=0}^m \begin{pmatrix} a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) & \mathbf{0} \\ b_{2n} + b_{2n+2}(-\mathbf{i} + \mathbf{j}) & a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) \end{pmatrix} t^{2n} \right. \\ &\quad \left. + \begin{pmatrix} a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & \mathbf{0} \\ b_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) \end{pmatrix} t^{2n+1} \right. \\ &\quad \left. \mid a_{2n}, b_{2n}, a_{2n+1}, b_{2n+1}, a_{2n+2}, b_{2n+2} \in \mathbb{R} \right\} \subseteq Z(M[t; \sigma]).\end{aligned}$$

Then we continue by showing that

$$\begin{aligned}
Z(M[t; \sigma]) \subseteq & \left\{ \sum_{n=0}^m \begin{pmatrix} a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) & \mathbf{0} \\ b_{2n} + b_{2n+2}(-\mathbf{i} + \mathbf{j}) & a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) \end{pmatrix} t^{2n} \right. \\
& + \left. \begin{pmatrix} a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & \mathbf{0} \\ b_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) \end{pmatrix} t^{2n+1} \right. \\
& \left. | a_{2n}, b_{2n}, a_{2n+1}, b_{2n+1}, a_{2n+2}, b_{2n+2} \in \mathbb{R} \right\}.
\end{aligned}$$

(1) We show that

$$\begin{aligned}
& \left\{ \sum_{n=0}^m \begin{pmatrix} a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) & \mathbf{0} \\ b_{2n} + b_{2n+2}(-\mathbf{i} + \mathbf{j}) & a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) \end{pmatrix} t^{2n} \right. \\
& + \left. \begin{pmatrix} a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & \mathbf{0} \\ b_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) \end{pmatrix} t^{2n+1} \right. \\
& \left. | a_{2n}, b_{2n}, a_{2n+1}, b_{2n+1}, a_{2n+2}, b_{2n+2} \in \mathbb{R} \right\} \subseteq Z(M[t; \sigma]).
\end{aligned}$$

Let

$$\begin{aligned}
p(t) \in & \left\{ \sum_{n=0}^m \begin{pmatrix} a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) & \mathbf{0} \\ b_{2n} + b_{2n+2}(-\mathbf{i} + \mathbf{j}) & a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) \end{pmatrix} t^{2n} \right. \\
& + \left. \begin{pmatrix} a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & \mathbf{0} \\ b_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) \end{pmatrix} t^{2n+1} \right. \\
& \left. | a_{2n}, b_{2n}, a_{2n+1}, b_{2n+1}, a_{2n+2}, b_{2n+2} \in \mathbb{R} \right\}.
\end{aligned}$$

Assume

$$\begin{aligned}
p(t) = & \sum_{n=0}^m \begin{pmatrix} g_{2n} + g_{2n+2}(-\mathbf{i} + \mathbf{j}) & \mathbf{0} \\ h_{2n} + h_{2n+2}(-\mathbf{i} + \mathbf{j}) & g_{2n} + g_{2n+2}(-\mathbf{i} + \mathbf{j}) \end{pmatrix} t^{2n} \\
& + \begin{pmatrix} g_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & \mathbf{0} \\ h_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & g_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) \end{pmatrix} t^{2n+1}.
\end{aligned}$$

We show that $p(t)q(t) = q(t)p(t)$, $\forall q(t) \in M[t; \sigma]$. Without loss of generality, the proof is divided into two cases:

$$q(t) = \begin{pmatrix} u_0 + u_1\mathbf{i} + u_2\mathbf{j} & \mathbf{0} \\ v_0 + v_1\mathbf{i} + v_2\mathbf{j} & u_0 + u_1\mathbf{i} + u_2\mathbf{j} \end{pmatrix} \text{ and } q(t) = t.$$

In the first case

$$q(t) = \begin{pmatrix} u_0 + u_1\mathbf{i} + u_2\mathbf{j} & \mathbf{0} \\ v_0 + v_1\mathbf{i} + v_2\mathbf{j} & u_0 + u_1\mathbf{i} + u_2\mathbf{j} \end{pmatrix}.$$

Therefore,

$$\begin{aligned} p(t)q(t) &= \left[\sum_{n=0}^m \begin{pmatrix} g_{2n} + g_{2n+2}(-\mathbf{i} + \mathbf{j}) & \mathbf{0} \\ h_{2n} + h_{2n+2}(-\mathbf{i} + \mathbf{j}) & g_{2n} + g_{2n+2}(-\mathbf{i} + \mathbf{j}) \end{pmatrix} t^{2n} \right. \\ &\quad \left. + \begin{pmatrix} g_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & \mathbf{0} \\ h_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & g_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) \end{pmatrix} t^{2n+1} \right] \\ &\cdot \begin{pmatrix} u_0 + u_1\mathbf{i} + u_2\mathbf{j} & \mathbf{0} \\ v_0 + v_1\mathbf{i} + v_2\mathbf{j} & u_0 + u_1\mathbf{i} + u_2\mathbf{j} \end{pmatrix}. \end{aligned}$$

Because $\sigma(\mathbf{a}) = a_0 - a_2\mathbf{i} - a_1\mathbf{j}$ and $\sigma^2(\mathbf{a}) = a_0 + a_1\mathbf{i} + a_2\mathbf{j}$, we get

$$\begin{aligned} &\sum_{n=0}^m \begin{pmatrix} (g_{2n} + g_{2n+2}(-\mathbf{i} + \mathbf{j}))(u_0 + u_1\mathbf{i} + u_2\mathbf{j}) \\ ((h_{2n} + h_{2n+2}(-\mathbf{i} + \mathbf{j}))(u_0 + u_1\mathbf{i} + u_2\mathbf{j}) + (g_{2n} + g_{2n+2}(-\mathbf{i} + \mathbf{j}))(v_0 + v_1\mathbf{i} + v_2\mathbf{j})) \\ \mathbf{0} \\ (g_{2n} + g_{2n+2}(-\mathbf{i} + \mathbf{j}))(u_0 + u_1\mathbf{i} + u_2\mathbf{j}) \end{pmatrix} t^{2n} \\ &+ \begin{pmatrix} (g_{2n+1}(-1 + \mathbf{i} - \mathbf{j}))(u_0 - u_2\mathbf{i} - u_1\mathbf{j}) \\ ((h_{2n+1}(-1 + \mathbf{i} - \mathbf{j}))(u_0 - u_2\mathbf{i} - u_1\mathbf{j}) + (g_{2n+1}(-1 + \mathbf{i} - \mathbf{j}))(v_0 - v_2\mathbf{i} - v_1\mathbf{j})) \\ \mathbf{0} \\ (g_{2n+1}(-1 + \mathbf{i} - \mathbf{j}))(u_0 - u_2\mathbf{i} - u_1\mathbf{j}) \end{pmatrix} t^{2n+1} \\ &= \sum_{n=0}^m \begin{pmatrix} c_{11} & \mathbf{0} \\ c_{21} & c_{22} \end{pmatrix} t^{2n} + \begin{pmatrix} d_{11} & \mathbf{0} \\ d_{21} & d_{22} \end{pmatrix} t^{2n+1}, \end{aligned}$$

where

$$\begin{aligned}
c_{11} &= (g_{2n}u_0 + g_{2n+2}(u_2 - u_1)) + (g_{2n}u_1 + g_{2n+2}(-u_0 - u_2))\mathbf{i} \\
&\quad + (g_{2n}u_2 + g_{2n+2}(u_1 + u_0))\mathbf{j}, \\
c_{21} &= (h_{2n}u_0 + h_{2n+2}(u_2 - u_1) + g_{2n}v_0 + g_{2n+2}(v_2 - v_1)) \\
&\quad + (h_{2n}u_1 - h_{2n+2}(u_0 + u_2) + g_{2n}v_1 - g_{2n+2}(v_0 + v_2))\mathbf{i} \\
&\quad + (h_{2n}u_2 + h_{2n+2}(u_1 + u_0) + g_{2n}v_2 + g_{2n+2}(v_1 + v_0))\mathbf{j}, \\
c_{22} &= (g_{2n}u_0 + g_{2n+2}(u_2 - u_1)) + (g_{2n}u_1 + g_{2n+2}(-u_0 - u_2))\mathbf{i} \\
&\quad + (g_{2n}u_2 + g_{2n+2}(u_1 + u_0))\mathbf{j}, \\
d_{11} &= (g_{2n+1}(-u_0 + u_1 - u_2)) + (g_{2n+1}(u_0 - u_1 + u_2))\mathbf{i} \\
&\quad + (g_{2n+1}(-u_0 + u_1 - u_2))\mathbf{j}, \\
d_{21} &= (h_{2n+1}(-u_0 + u_1 - u_2) + g_{2n+1}(-v_0 + v_1 - v_2)) \\
&\quad + (h_{2n+1}(u_0 - u_1 + u_2) + g_{2n+1}(v_0 - v_1 + v_2))\mathbf{i} \\
&\quad + (h_{2n+1}(-u_0 + u_1 - u_2) + g_{2n+1}(-v_0 + v_1 - v_2))\mathbf{j}, \\
d_{22} &= (g_{2n+1}(-u_0 + u_1 - u_2)) + (g_{2n+1}(u_0 - u_1 + u_2))\mathbf{i} \\
&\quad + (g_{2n+1}(-u_0 + u_1 - u_2))\mathbf{j}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&q(t)p(t) \\
&= \begin{pmatrix} u_0 + u_1\mathbf{i} + u_2\mathbf{j} & \mathbf{0} \\ v_0 + v_1\mathbf{i} + v_2\mathbf{j} & u_0 + u_1\mathbf{i} + u_2\mathbf{j} \end{pmatrix} \\
&\cdot \left[\sum_{n=0}^m \begin{pmatrix} g_{2n} + g_{2n+2}(-\mathbf{i} + \mathbf{j}) & \mathbf{0} \\ h_{2n} + h_{2n+2}(-\mathbf{i} + \mathbf{j}) & g_{2n} + g_{2n+2}(-\mathbf{i} + \mathbf{j}) \end{pmatrix} t^{2n} \right. \\
&\left. + \begin{pmatrix} g_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & \mathbf{0} \\ h_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & g_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) \end{pmatrix} t^{2n+1} \right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^m \left(\begin{array}{c} (u_0 + u_1 \mathbf{i} + u_2 \mathbf{j})(g_{2n} + g_{2n+2}(-\mathbf{i} + \mathbf{j})) \\ (v_0 + v_1 \mathbf{i} + v_2 \mathbf{j})(g_{2n} + g_{2n+2}(-\mathbf{i} + \mathbf{j})) + (u_0 + u_1 \mathbf{i} + u_2 \mathbf{j})(h_{2n} + h_{2n+2}(-\mathbf{i} + \mathbf{j})) \end{array} \right) t^{2n} \\
&\quad + \left(\begin{array}{c} (u_0 + u_1 \mathbf{i} + u_2 \mathbf{j})g_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) \\ (v_0 + v_1 \mathbf{i} + v_2 \mathbf{j})g_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) + (u_0 + u_1 \mathbf{i} + u_2 \mathbf{j})h_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) \end{array} \right) t^{2n+1} \\
&= \sum_{n=0}^m \begin{pmatrix} e_{11} & \mathbf{0} \\ e_{21} & e_{22} \end{pmatrix} t^{2n} + \begin{pmatrix} f_{11} & \mathbf{0} \\ f_{21} & f_{22} \end{pmatrix} t^{2n+1},
\end{aligned}$$

where

$$\begin{aligned}
e_{11} &= (g_{2n}u_0 + g_{2n+2}(u_2 - u_1)) + (g_{2n}u_1 + g_{2n+2}(-u_0 - u_2))\mathbf{i} \\
&\quad + (g_{2n}u_2 + g_{2n+2}(u_1 + u_0))\mathbf{j},
\end{aligned}$$

$$\begin{aligned}
e_{21} &= (h_{2n}u_0 + h_{2n+2}(u_2 - u_1) + g_{2n}v_0 + g_{2n+2}(v_2 - v_1)) \\
&\quad + (h_{2n}u_1 - h_{2n+2}(u_0 + u_2) + g_{2n}v_1 - g_{2n+2}(v_0 + v_2))\mathbf{i} \\
&\quad + (h_{2n}u_2 + h_{2n+2}(u_1 + u_0) + g_{2n}v_2 + g_{2n+2}(v_1 + v_0))\mathbf{j},
\end{aligned}$$

$$\begin{aligned}
e_{22} &= (g_{2n}u_0 + g_{2n+2}(u_2 - u_1)) + (g_{2n}u_1 + g_{2n+2}(-u_0 - u_2))\mathbf{i} \\
&\quad + (g_{2n}u_2 + g_{2n+2}(u_1 + u_0))\mathbf{j},
\end{aligned}$$

$$\begin{aligned}
f_{11} &= (g_{2n+1}(-u_0 + u_1 - u_2)) + (g_{2n+1}(u_0 - u_1 + u_2))\mathbf{i} \\
&\quad + (g_{2n+1}(-u_0 + u_1 - u_2))\mathbf{j},
\end{aligned}$$

$$\begin{aligned}
f_{21} &= (h_{2n+1}(-u_0 + u_1 - u_2) + g_{2n+1}(-v_0 + v_1 - v_2)) \\
&\quad + (h_{2n+1}(u_0 - u_1 + u_2) + g_{2n+1}(v_0 - v_1 + v_2))\mathbf{i} \\
&\quad + (h_{2n+1}(-u_0 + u_1 - u_2) + g_{2n+1}(-v_0 + v_1 - v_2))\mathbf{j},
\end{aligned}$$

$$\begin{aligned} f_{22} = & (g_{2n+1}(-u_0 + u_1 - u_2)) + (g_{2n+1}(u_0 - u_1 + u_2))\mathbf{i} \\ & + (g_{2n+1}(-u_0 + u_1 - u_2))\mathbf{j}. \end{aligned}$$

We can see that $p(t)q(t) = q(t)p(t)$. Therefore, $p(t) \in Z(M[t; \sigma])$.

(2) We show that

$$\begin{aligned} Z(M[t; \sigma]) \subseteq & \left\{ \sum_{n=0}^m \begin{pmatrix} a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) & \mathbf{0} \\ b_{2n} + b_{2n+2}(-\mathbf{i} + \mathbf{j}) & a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) \end{pmatrix} t^{2n} \right. \\ & + \left. \begin{pmatrix} a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & \mathbf{0} \\ b_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) \end{pmatrix} t^{2n+1} \right. \\ & \left. | a_{2n}, b_{2n}, a_{2n+1}, b_{2n+1}, a_{2n+2}, b_{2n+2} \in \mathbb{R} \right\}. \end{aligned}$$

Let $p(t) \in Z(M[t; \sigma])$. Then

$$p(t)q(t) = q(t)p(t), \quad \forall q(t) \in M[t; \sigma].$$

$$\begin{aligned} p(t) = & \sum_{n=0}^m \left(\begin{bmatrix} g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} & \mathbf{0} \\ h_0^{2n} + h_1^{2n}\mathbf{i} + h_2^{2n}\mathbf{j} & g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} \end{bmatrix} t^{2n} \right. \\ & \left. + \begin{bmatrix} g_0^{2n+1} + g_1^{2n+1}\mathbf{i} + g_2^{2n+1}\mathbf{j} & \mathbf{0} \\ h_0^{2n+1} + h_1^{2n+1}\mathbf{i} + h_2^{2n+1}\mathbf{j} & g_0^{2n+1} + g_1^{2n+1}\mathbf{i} + g_2^{2n+1}\mathbf{j} \end{bmatrix} t^{2n+1} \right). \end{aligned}$$

By choosing $q(t) = \begin{pmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{i} & \mathbf{i} \end{pmatrix}$,

$$\begin{aligned} & p(t)q(t) \\ = & \sum_{n=0}^m \left(\begin{bmatrix} g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} & \mathbf{0} \\ h_0^{2n} + h_1^{2n}\mathbf{i} + h_2^{2n}\mathbf{j} & g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} \end{bmatrix} t^{2n} \right. \\ & \left. + \begin{bmatrix} g_0^{2n+1} + g_1^{2n+1}\mathbf{i} + g_2^{2n+1}\mathbf{j} & \mathbf{0} \\ h_0^{2n+1} + h_1^{2n+1}\mathbf{i} + h_2^{2n+1}\mathbf{j} & g_0^{2n+1} + g_1^{2n+1}\mathbf{i} + g_2^{2n+1}\mathbf{j} \end{bmatrix} t^{2n+1} \right) \begin{pmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{i} & \mathbf{i} \end{pmatrix}. \end{aligned}$$

Because $\sigma(\mathbf{i}) = -\mathbf{j}$ and $\sigma^2(\mathbf{i}) = \mathbf{i}$, we have

$$\begin{aligned}
 p(t)q(t) &= \sum_{n=0}^m \left[\begin{array}{c} -g_2^{2n} + g_0^{2n}\mathbf{i} + g_1^{2n}\mathbf{j} \\ -h_2^{2n} + h_0^{2n}\mathbf{i} + h_1^{2n}\mathbf{j} - g_2^{2n} + g_0^{2n}\mathbf{i} + g_1^{2n}\mathbf{j} \\ \mathbf{0} \\ -g_2^{2n} + g_0^{2n}\mathbf{i} + g_1^{2n}\mathbf{j} \end{array} \right] t^{2n} \\
 &\quad - \left[\begin{array}{c} -g_1^{2n+1} - g_2^{2n+1}\mathbf{i} + g_0^{2n+1}\mathbf{j} \\ -h_1^{2n+1} - h_2^{2n+1}\mathbf{i} + h_0^{2n+1}\mathbf{j} - g_1^{2n+1} - g_2^{2n+1}\mathbf{i} + g_0^{2n+1}\mathbf{j} \\ \mathbf{0} \\ -g_1^{2n+1} - g_2^{2n+1}\mathbf{i} + g_0^{2n+1}\mathbf{j} \end{array} \right] t^{2n+1}.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 q(t)p(t) &= \begin{pmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{i} & \mathbf{i} \end{pmatrix} \left(\sum_{n=0}^m \left[\begin{array}{cc} g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} & \mathbf{0} \\ h_0^{2n} + h_1^{2n}\mathbf{i} + h_2^{2n}\mathbf{j} & g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} \end{array} \right] t^{2n} \right. \\
 &\quad \left. + \left[\begin{array}{cc} g_0^{2n+1} + g_1^{2n+1}\mathbf{i} + g_2^{2n+1}\mathbf{j} & \mathbf{0} \\ h_0^{2n+1} + h_1^{2n+1}\mathbf{i} + h_2^{2n+1}\mathbf{j} & g_0^{2n+1} + g_1^{2n+1}\mathbf{i} + g_2^{2n+1}\mathbf{j} \end{array} \right] t^{2n+1} \right) \\
 &= \sum_{n=0}^m \left[\begin{array}{c} -g_2^{2n} + g_0^{2n}\mathbf{i} + g_1^{2n}\mathbf{j} \\ -h_2^{2n} + h_0^{2n}\mathbf{i} + h_1^{2n}\mathbf{j} - g_2^{2n} + g_0^{2n}\mathbf{i} + g_1^{2n}\mathbf{j} \\ \mathbf{0} \\ -g_2^{2n} + g_0^{2n}\mathbf{i} + g_1^{2n}\mathbf{j} \end{array} \right] t^{2n} \\
 &\quad + \left[\begin{array}{c} -g_2^{2n+1} + g_0^{2n+1}\mathbf{i} + g_1^{2n+1}\mathbf{j} \\ -h_2^{2n+1} + h_0^{2n+1}\mathbf{i} + h_1^{2n+1}\mathbf{j} - g_2^{2n+1} + g_0^{2n+1}\mathbf{i} + g_1^{2n+1}\mathbf{j} \\ \mathbf{0} \\ -g_2^{2n+1} + g_0^{2n+1}\mathbf{i} + g_1^{2n+1}\mathbf{j} \end{array} \right] t^{2n+1}.
 \end{aligned}$$

Because $p_{2n}\mathbf{i} = \mathbf{i}p_{2n}$ and $-p_{2n+1}\mathbf{j} = \mathbf{i}p_{2n+1}$, we get

$$p_{2n+1} = \begin{bmatrix} g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & \mathbf{0} \\ h_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) \end{bmatrix}.$$

Then we conclude that

$$\begin{aligned} p(t) &= \sum_{n=0}^m (p_{2n}t^{2n} + p_{2n+1}t^{2n+1}) \\ &= \sum_{n=0}^m \begin{bmatrix} g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} & \mathbf{0} \\ h_0^{2n} + h_1^{2n}\mathbf{i} + h_2^{2n}\mathbf{j} & g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} \end{bmatrix} t^{2n} \\ &\quad + \begin{bmatrix} g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & \mathbf{0} \\ h_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) \end{bmatrix} t^{2n+1}. \end{aligned}$$

Next, choose $q(t) = t$. Then

$$\begin{aligned} p(t)q(t) &= \left(\sum_{n=0}^m \begin{bmatrix} g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} & \mathbf{0} \\ h_0^{2n} + h_1^{2n}\mathbf{i} + h_2^{2n}\mathbf{j} & g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} \end{bmatrix} t^{2n} \right. \\ &\quad \left. + \begin{bmatrix} g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & \mathbf{0} \\ h_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) \end{bmatrix} t^{2n+1} \right) t \\ &= \sum_{n=0}^m \begin{bmatrix} g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} & \mathbf{0} \\ h_0^{2n} + h_1^{2n}\mathbf{i} + h_2^{2n}\mathbf{j} & g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} \end{bmatrix} t^{2n+1} \\ &\quad + \begin{bmatrix} g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & \mathbf{0} \\ h_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) \end{bmatrix} t^{2n+2}. \end{aligned}$$

On the other hand,

$$q(t)p(t) = t \left(\sum_{n=0}^m \begin{bmatrix} g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} & \mathbf{0} \\ h_0^{2n} + h_1^{2n}\mathbf{i} + h_2^{2n}\mathbf{j} & g_0^{2n} + g_1^{2n}\mathbf{i} + g_2^{2n}\mathbf{j} \end{bmatrix} t^{2n} \right)$$

$$+ \begin{bmatrix} g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & \mathbf{0} \\ h_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) \end{bmatrix} t^{2n+1}.$$

Because $t\mathbf{i} = \sigma(\mathbf{i})t = -\mathbf{j}t$ and $t\mathbf{j} = \sigma(\mathbf{j})t = -\mathbf{i}t$, we have

$$\begin{aligned} q(t)p(t) &= \sum_{n=0}^m \begin{bmatrix} g_0^{2n} - g_2^{2n}\mathbf{i} - g_1^{2n}\mathbf{j} & \mathbf{0} \\ h_0^{2n} - h_2^{2n}\mathbf{i} - h_1^{2n}\mathbf{j} & g_0^{2n} - g_2^{2n}\mathbf{i} - g_1^{2n}\mathbf{j} \end{bmatrix} t^{2n+1} \\ &\quad + \begin{bmatrix} g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & \mathbf{0} \\ h_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) \end{bmatrix} t^{2n+2}. \end{aligned}$$

Because $p(t)q(t) = q(t)p(t)$, we get $g_1^{2n} = -g_2^{2n}$ and $h_1^{2n} = -h_2^{2n}$.

Therefore,

$$\begin{aligned} p(t) &= \sum_{n=0}^m \begin{bmatrix} g_0^{2n} + g_2^{2n}(-\mathbf{i} + \mathbf{j}) & \mathbf{0} \\ h_0^{2n} + h_2^{2n}(-\mathbf{i} + \mathbf{j}) & g_0^{2n} + g_2^{2n}(-\mathbf{i} + \mathbf{j}) \end{bmatrix} t^{2n} \\ &\quad + \begin{bmatrix} g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & \mathbf{0} \\ h_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & g_1^{2n+1}(-1 + \mathbf{i} + \mathbf{j}) \end{bmatrix} t^{2n+1}. \end{aligned}$$

Now, $p(t)$ in general can be written as:

$$\begin{aligned} p(t) &= \sum_{n=0}^m \begin{bmatrix} g_{2n} + g_{2n+2}(-\mathbf{i} + \mathbf{j}) & \mathbf{0} \\ h_{2n} + h_{2n+2}(-\mathbf{i} + \mathbf{j}) & g_{2n} + g_{2n+2}(-\mathbf{i} + \mathbf{j}) \end{bmatrix} t^{2n} \\ &\quad + \begin{bmatrix} g_{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & \mathbf{0} \\ h_{2n+1}(-1 + \mathbf{i} + \mathbf{j}) & g_{2n+1}(-1 + \mathbf{i} + \mathbf{j}) \end{bmatrix} t^{2n+1} \\ p(t) &= \sum_{n=0}^m (p_{2n}t^{2n} + p_{2n+1}t^{2n+1}). \end{aligned}$$

Therefore,

$$p(t) \in \left\{ \sum_{n=0}^m \begin{pmatrix} a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) & \mathbf{0} \\ b_{2n} + b_{2n+2}(-\mathbf{i} + \mathbf{j}) & a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) \end{pmatrix} t^{2n} \right. \\ \left. + \begin{pmatrix} a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & \mathbf{0} \\ b_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) \end{pmatrix} t^{2n+1} \right\}.$$

Thus,

$$Z(M[t; \sigma]) \subseteq \left\{ \sum_{n=0}^m \begin{pmatrix} a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) & \mathbf{0} \\ b_{2n} + b_{2n+2}(-\mathbf{i} + \mathbf{j}) & a_{2n} + a_{2n+2}(-\mathbf{i} + \mathbf{j}) \end{pmatrix} t^{2n} \right. \\ \left. + \begin{pmatrix} a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & \mathbf{0} \\ b_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) & a_{2n+1}(-1 + \mathbf{i} - \mathbf{j}) \end{pmatrix} t^{2n+1} \right\}.$$

□

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