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LAMPIRAN

A. Perhitungan Persamaan Geodesik

Berangkat dari persamaan (5):

$$I = \int \mathcal{L}\ ds \tag{A.1}$$

Kemudian divariasikan:

$$\delta I = \delta \int \mathcal{L}\ ds = \int \delta\mathcal{L}\, ds \tag{A.2}$$

Mengingat bahwa $\mathcal{L} = \left(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)^{\frac{1}{2}}$, maka variasi dari $\mathcal{L}$:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x^\mu}\ \delta x^\mu + \frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\delta\left(\frac{dx^\mu}{ds}\right)\ ;\ \delta\left(\frac{dx^\mu}{ds}\right) = \frac{d}{ds}\delta x^\mu$$

$$= \frac{\partial\mathcal{L}}{\partial x^\mu}\ \delta x^\mu + \frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\frac{d}{ds}\delta x^\mu$$

$$= \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{d}{ds}\left[\frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\delta x^\mu\right] - \left[\frac{d}{ds}\frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\right]\delta x^\mu\ ds \tag{A.3}$$

Variasi δx^μ pada batas titik awal dan titik akhir adalah nol, maka suku kedua pada ruas kanan di atas menjadi lenyap.

$$\delta\int\mathcal{L}\ ds = \int\left[\frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{d}{ds}\frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\right]\ \delta x^\mu\ ds = 0 \tag{A.4}$$

Karena $\delta x^\mu\ ds$ bernilai apa saja, maka persamaan geodesik ditentukan oleh:

$$\frac{\partial\mathcal{L}}{\partial x^\mu} - \left[\frac{d}{ds}\frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}\right] = 0 \tag{A.5}$$

Sekarang selesaikan bentuk $\frac{\partial\mathcal{L}}{\partial x^\mu}$ dan $\frac{d}{ds}\frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)}$, dengan memasukkan nilai $\mathcal{L}$ dan $ds^2 = (g_{\rho\sigma}dx^\rho dx^\sigma)$, diperoleh hubungan:

$$\frac{\partial\mathcal{L}}{\partial x^\mu} = \frac{1}{2}\left(g_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds}\right)^{-\frac{1}{2}}\ \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}$$

$$= \frac{1}{2}\frac{ds}{ds}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{ds}\frac{ds^\beta}{ds}$$

$$= \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}$$

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)} = \left(g_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds}\right)^{-\frac{1}{2}}\ g_{\mu\alpha}\frac{dx^\alpha}{ds}$$

$$= \frac{ds}{ds}\ g_{\mu\alpha}\frac{dx^\alpha}{ds} = g_{\mu\alpha}\frac{dx^\alpha}{ds}$$

$$\frac{d}{ds}\frac{\partial \mathcal{L}}{\partial\left(\frac{dx^\mu}{ds}\right)} = \frac{ds}{ds}\left(g_{\mu\alpha}\frac{dx^\alpha}{ds}\right)$$

$$= g_{\mu\alpha}\frac{d^2x^\alpha}{ds^2} + \frac{\partial g_{\mu\alpha}}{\partial x^\beta}\frac{dx^\alpha}{ds}\frac{ds^\beta}{ds}$$

Subtitusi hasil ke persamaan (A.5) diperoleh:

$$\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - g_{\mu\alpha}\frac{d^2x^\alpha}{ds^2} - \frac{\partial g_{\mu\alpha}}{\partial x^\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$$

$$g_{\mu\alpha}\frac{d^2x^\alpha}{ds^2} + \frac{1}{2}\left(2\frac{\partial g_{\mu\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right)\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$$

$$g_{\mu\alpha}\frac{d^2x^\alpha}{ds^2} + \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right)\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$$

simbol Christoffel jenis pertama:

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right) \qquad \text{(A.6)}$$

Sehingga persamaan geodesik dapat dituliskan sebagai berikut:

$$g_{\mu\nu}\frac{d^2x^\alpha}{ds^2} + \Gamma_{\alpha\beta\mu}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$$

$$\frac{d^2x^\gamma}{ds^2} + \Gamma^\gamma_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \qquad \text{(A.7)}$$

B. Perhitungan Geodesik dalam Bidang Ekuator

1. Metrik Kerr dalam Koordinat Boyer-Linquist di Ekuator

Berangkat dari persamaan (3) dimana

$$ds^2 = \frac{-\Delta - a^2 sin^2\theta}{\rho^2}dt^2 - 2\frac{2Marsin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$
$$+ \frac{(r^2+a^2) - a^2\Delta sin^2\theta}{\rho^2}sin^2\theta d\phi^2 \qquad \text{(B.1)}$$

Ditinjau dari ekuator $\theta = \frac{\pi}{2}$, $\Delta = r^2 - 2Mr + a^2$ dan $\rho^2 = r^2 + a^2cos^2\theta$

$$ds^2 = \frac{-[(r^2-2Mr+a^2)-a^2]}{r^2}dt^2 - 2\frac{2Ma}{r}dtd\phi + \frac{r^2}{r^2-2Mr+a^2}dr^2 + r^2d\theta^2 + \frac{(r^2+a^2)^2 - a^2(r^2-2Mr+a^2)}{r^2}d\phi^2$$

$$= 1\left(1 - \frac{2M}{r}\right)dt^2 - 2\frac{2Ma}{r}dtd\phi + \frac{r^2}{\Delta}dr^2 + \frac{(r^2+a^2)^2 - a^2(r^2-2Mr+a^2)}{r^2}d\phi^2$$

Uraikan $(r^2+a^2)^2$ dan $a^2(r^2-2Mr+a^2)$:

$$(r^2+a^2)^2 = r^4 + a^2r^2 + a^2r^2 + a^4 = r^4 + 2a^2r^2 + a^4$$

$$a^2(r^2 - 2Mr + a^2) = a^2r^2 - 2Ma^2r + a^4$$

menghasilkan:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 - 2\frac{2Ma}{r}dtd\phi + \frac{r^2}{\Delta}dr^2 + \left[(r^2+a^2) + \frac{2Ma^2}{r}\right]d\phi^2 \qquad \text{(B.2)}$$

2. Mencari Komponen Metrik Kontravarian

Komponen metrik kontravarian didapatkan dari menghitung invers metrik kovarian:

$$g^{\mu\nu} = \frac{1}{\det(g_{\mu\nu})} adj\left(g_{\mu\nu}\right) \qquad \text{(B.3)}$$

Karena elemen diagonal tidak bergantung pada t dan ϕ maka:

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix} \rightarrow g^{\mu\nu} = \frac{1}{|g_{\mu\nu}|}\begin{pmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{t\phi} & g_{tt} \end{pmatrix} \qquad \text{(B.4)}$$

Komponen metrik kovarian didapatkan dari persamaan (B.2) dengan meninjau bentuk umum metrik aksial stationer (1). Menghitung determinan $g_{\mu\nu}$:

$$\begin{aligned}
|g_{\mu\nu}| &= g_{tt}g_{\phi\phi} - \left(g_{t\phi}\right)^2 \\
&= -\left(1-\frac{2M}{r}\right)\left[(r^2+a^2)+\frac{2Ma^2}{r}\right] - \left(-\frac{2Ma}{r}\right)^2 \\
&= -\left[\frac{2Ma^2}{r} + (r^2+a^2) - \frac{2M}{r}(r^2+a^2) - \frac{4M^2a^2}{r^2}\right] - \frac{4M^2a^2}{r^2} \\
&= -\frac{2Ma^2}{r} - (r^2-a^2) + 2Mr + \frac{2Ma^2}{r} + \frac{4M^2a^2}{r^2} - \frac{4M^2a^2}{r^2} \\
&= -(r^2+a^2-2Mr) = -\Delta
\end{aligned}$$

Maka komponen metrik kontravarian $g^{\mu\nu}$:

$$g^{tt} = \frac{1}{|g_{\mu\nu}|}(g_{\phi\phi}) = -\frac{1}{\Delta}\left[(r^2+a^2)+\frac{2Ma^2}{r}\right] \qquad \text{(B.5)}$$

$$g^{t\phi} = \frac{1}{|g_{\mu\nu}|}\left(-g_{t\phi}\right) = -\frac{1}{\Delta}\left(-\frac{2Ma}{r}\right) = \frac{2Ma}{r} \qquad \text{(B.6)}$$

$$g^{t\phi} = \frac{1}{|g_{\mu\nu}|}(g_{tt}) = -\frac{1}{\Delta}\left[-\left(1-\frac{2M}{r}\right)\right] = \frac{1}{\Delta}\left(1-\frac{2M}{r}\right) \qquad \text{(B.7)}$$

$$g^{rr} = \frac{1}{g_{rr}} = \frac{\Delta}{r^2} \qquad \text{(B.8)}$$

$$g^{\theta\theta} = \frac{1}{g_{\theta\theta}} = \frac{1}{r^2} \qquad \text{(B.9)}$$

3. Mencari Komponen Kecepatan Koordinat $\dot{t}$, $\dot{\phi}$, dan $\dot{r}$

Komponen kecepatan koordinat $\dot{t}$, $\dot{\phi}$, dan $\dot{r}$ dalam geodesik partikel di sekitar lubang hitam Kerr dapat dihitung menggunakan persamaan Euler-Lagrange dari Lagrangian relativistik:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \quad \text{(B.10)}$$

Dengan persamaan Lagrange:

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^{\mu}}\right) - \frac{\partial \mathcal{L}}{\partial x^{\mu}} = 0 \quad \text{(B.11)}$$

Untuk E:

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) - \frac{\partial \mathcal{L}}{\partial t} = 0$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = \frac{\partial \mathcal{L}}{\partial t}$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial}{\partial \dot{t}}\frac{1}{2}\left(g_{tt}\dot{t}^2 + 2g_{t\phi}\dot{t}\dot{\phi} + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2\right)\right) = 0$$

$$\frac{\partial}{\partial\lambda}(g_{tt}\dot{t} + g_{t\phi}\dot{\phi}) = 0$$

$$g_{tt}\dot{t} + g_{t\phi}\dot{\phi} = -E$$

$$-\left(1 - \frac{2M}{r}\right)\dot{t} + \left(-\frac{2Ma}{r}\right)\dot{\phi} = -E \quad \text{(B.12)}$$

Untuk J:

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial}{\partial \dot{\phi}}\frac{1}{2}\left(g_{tt}\dot{t}^2 + 2g_{t\phi}\dot{t}\dot{\phi} + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2\right)\right) = 0$$

$$\frac{\partial}{\partial\lambda}(g_{\phi\phi}\dot{\phi} + g_{t\phi}\dot{t}) = 0$$

$$g_{\phi\phi}\dot{\phi} + g_{t\phi}\dot{t} = J$$

$$\left[(r^2 + a^2) + \frac{2Ma^2}{r}\right]\dot{\phi} + \left(-\frac{2Ma}{r}\right)\dot{t} = J \quad \text{(B.13)}$$

Kemudian eliminasi untuk $\dot{t}$ dan $\dot{\phi}$

Komponen $\dot{t}$:

$$-\left(1-\frac{2M}{r}\right)\dot{t}+\left(-\frac{2Ma}{r}\right)\dot{\phi}=-E$$

$$\left(-\frac{2Ma}{r}\right)\dot{\phi}=-E+\left(1-\frac{2M}{r}\right)\dot{t}$$

$$\dot{\phi}=\left[-E+\left(1-\frac{2M}{r}\right)\dot{t}\right]\left(-\frac{r}{2Ma}\right)$$

Subtitusi nilai $\dot{\phi}$ ke persamaan (B.13)

$$\left[(r^2+a^2)+\frac{2Ma^2}{r}\right]\dot{\phi}+\left(-\frac{2Ma}{r}\right)\dot{t}=J$$

$$\left[(r^2+a^2)+\frac{2Ma^2}{r}\right]\left[-E+\left(1-\frac{2M}{r}\right)\dot{t}\right]\left(-\frac{r}{2Ma}\right)+\left(-\frac{2Ma}{r}\right)\dot{t}=J$$

$$-\frac{2Ma}{r}\dot{t}-\frac{r}{2Ma}\left[-E(r^2+a^2)-\frac{2Ma^2}{r}E+(r^2+a^2)\left(1-\frac{2M}{r}\right)\dot{t}+\frac{2Ma^2}{r}\left(1-\frac{2M}{r}\right)\dot{t}\right]=J$$

$$-\frac{2Ma}{r}\dot{t}+\frac{r(r^2+a^2)E}{2Ma}+aE-\frac{r}{2Ma}(r^2+a^2)\left(1-\frac{2M}{r}\right)\dot{t}-a\left(1-\frac{2M}{r}\right)\dot{t}=J$$

$$\dot{t}\left[-\frac{2Ma}{r}-\frac{r}{2Ma}(r^2+a^2)\left(1-\frac{2M}{r}\right)-a\left(1-\frac{2M}{r}\right)\right]=J-\frac{r(r^2+a^2)E}{2Ma}-aE$$

$$\dot{t}\left(-\frac{r}{2Ma}\Delta\right)=J-aE-\frac{r(r^2+a^2)E}{2Ma}$$

$$\dot{t}=-\frac{1}{\Delta}\left(\frac{2Ma}{r}J-\frac{2Ma^2E}{r}-(r^2+a^2)E\right)$$

$$\dot{t}=\frac{1}{\Delta}\left[\left((r^2+a^2)+\frac{2Ma^2}{r}\right)E-\frac{2Ma}{r}J\right] \qquad \text{(B.14)}$$

Komponen $\dot{\phi}$:

$$-\left(1-\frac{2M}{r}\right)\dot{t}+\left(-\frac{2Ma}{r}\right)\dot{\phi}=-E$$

$$-\left(1-\frac{2M}{r}\right)\dot{t}=-E+\left(\frac{2Ma}{r}\right)\dot{\phi}$$

$$\dot{t}=\left[-E+\left(\frac{2Ma}{r}\right)\dot{\phi}\right]\left(-1+\frac{r}{2M}\right)$$

$$\dot{t} = \left[-E + \left(\frac{2Ma}{r}\right)\dot{\phi}\right]\left(-\frac{r}{r-2M}\right)$$

Subtitusi nilai $\dot{t}$ ke persamaan (B.12)

$$\left[(r^2 + a^2) + \frac{2Ma^2}{r}\right]\dot{\phi} + \left(-\frac{2Ma}{r}\right)\dot{t} = J$$

$$\left[(r^2 + a^2) + \frac{2Ma^2}{r}\right]\dot{\phi} + \left(-\frac{2Ma}{r}\right)\left[-E + \frac{2Ma}{r}\dot{\phi}\right]\left(-\frac{r}{r-2M}\right) = J$$

$$\left[(r^2 + a^2) + \frac{2Ma^2}{r}\right]\dot{\phi} + \left[\frac{2MaE}{r} - \frac{4M^2a^2}{r^2}\dot{\phi}\right]\left(-\frac{r}{r-2M}\right) = J$$

$$\left[(r^2 + a^2) + \frac{2Ma^2}{r}\right]\dot{\phi} + \left[\left(-\frac{2MaE}{r-2M}\right) + \frac{4M^2a^2}{r(r-2M)}\dot{\phi}\right] = J$$

$$\dot{\phi}\left[(r^2 + a^2) + \frac{2Ma^2}{r} + \frac{4M^2a^2}{r(r-2M)}\right] = J + \frac{2MaE}{r-2M}$$

$$\dot{\phi}[r(r-2M)(r^2 + a^2) + 2Ma^2(r-2M) + 4M^2a^2] = r(r-2M)\left[J + \frac{2MaE}{r-2M}\right]$$

$$\dot{\phi}[r^4 + a^2r^2 - 2Mr^3 - 2Ma^2r + 2Ma^2r - 4M^2a^2 + 4M^2a^2]$$
$$= r(r-2M)J + 2MarE$$

$$\dot{\phi}[r^4 + a^2r^2 - 2Mr^3] = r(r-2M)J + 2MarE$$

$$\dot{\phi}r^2(r^2 + a^2 - 2Mr) = r(r-2M)J + 2MarE$$

$$\dot{\phi}r^2\Delta = r(r-2M)J + 2MarE$$

$$\dot{\phi} = \frac{1}{\Delta}\left(1 - \frac{2M}{r}\right)J + \frac{2MaE}{r} \qquad \text{(B.15)}$$

Komponen koordinat $\dot{r}$:

Dengan menggunakan $g^{\mu\nu}P_\mu P_\nu = \epsilon^2$

$$g^{tt}(P_t)^2 + 2g^{t\phi}P_tP_\phi + g^{\phi\phi}\left(P_\phi\right)^2 + g^{rr}(P_r)^2 = \epsilon^2$$

Subtitusi $P_t = -E$, $P_t = -E$ dan $P_r = g_{rr}\frac{d_r}{d_\lambda} = g_{rr}\dot{r}$

$$g^{tt}E^2 - 2g^{t\phi}EJ + g^{\phi\phi}J^2 + g_{rr}\dot{r} = -\epsilon^2$$

$$g_{rr}\dot{r} = -\epsilon^2 - g^{tt}E^2 + 2g^{t\phi}EJ - g^{\phi\phi}J^2$$

$$\dot{r} = \frac{1}{g_{rr}}[-\epsilon^2 - g^{tt}E^2 + 2g^{t\phi}EJ - g^{\phi\phi}J^2]$$

Kemudian subtitusi komponen-komponen metrik:

$$\dot{r}^2 = \frac{\Delta}{r^2}\left[\epsilon^2 + \frac{1}{\Delta}\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)E^2 + \frac{4MaEJ}{\Delta r} - \frac{1}{\Delta}\left(1 - \frac{2M}{r}\right)J^2\right]$$

$$= -\frac{\Delta\epsilon^2}{r^2} + \frac{1}{r^2}\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)E^2 + \frac{4MaEJ}{r^3} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)J^2$$

$$= -\frac{(r^2 - 2Mr + a^2)\epsilon^2}{r^2} + \frac{E^2}{r^2}\left(r^2 + a^2 + \frac{2Ma^2}{r}\right) + \frac{4MaEJ}{r^3} - \frac{J^2}{r^2}\left(1 - \frac{2M}{r}\right)$$

$$= \epsilon^2 + \frac{2M\epsilon^2}{r} - \frac{\epsilon^2 a^2}{r^2} + E^2 + \frac{2Ma^2E^2}{r^3} - \frac{4MaEJ}{r^2} - \frac{J^2}{r^2} + \frac{2MJ^2}{r^3}$$

$$= E^2 - \epsilon^2 + \frac{2M\epsilon^2}{r} + \frac{a^2(-\epsilon^2 + E^2) - J^2}{r^2} + \frac{2M(E^2a^2 - 2aEJ + J^2)}{r^3}$$

$$= E^2 - \epsilon^2 + \frac{2M\epsilon^2}{r} + \frac{a^2(-\epsilon^2 + E^2) - J^2}{r^2} + \frac{2M(aE - J)^2}{r^3} \quad \text{(B.16)}$$

4. Persamaan Lintasan Partikel Bermassa di Ekuator

Dengan meninjau kembali persamaan (B.16), untuk partikel bermassa ($\epsilon^2 = 1$), persamaan menjadi:

$$\dot{r}^2 = (E^2 - 1) + \frac{2M}{r} + \frac{a^2(E^2 - 1) - J^2}{r^2} + \frac{2M(aE - J)^2}{r^3} \quad \text{(B.17)}$$

Kemudian menyusun ulang persamaan, diperoleh:

$$\dot{r}^2 - \frac{2M}{r} - \frac{a^2(E^2 - 1) - J^2}{r^2} - \frac{2M(aE - J)^2}{r^3} = (E^2 - 1) \quad \text{(B.18)}$$

Selanjutnya bentuk persamaan dapat ditulis sebagai:

$$\frac{1}{2}\dot{r}^2 - \frac{M}{r} - \frac{a^2(E^2 - 1)J^2}{2r^2} - \frac{M(aE - J)^2}{r^3} = \frac{1}{2}(E^2 - 1) \quad \text{(B.19)}$$

Dengan potensial efektif (V_{eff}):

$$\left(V_{eff}(r, J, E) = -\frac{M}{r} + \frac{J^2 - a^2(E^2 - 1)}{2r^2} - \frac{M(aE - J)^2}{r^3}\right)$$

C. Gerak Melingkar Partikel Bermassa di Ekuator

Untuk gerak melingkar partikel dipilih $\dot{r} = 0$ dan $\ddot{r} = 0$. Tinjau potensial efektif:

$$V_{eff} = -\frac{M}{r} + \frac{J^2 - a^2(E^2 - 1)}{2r^2} - \frac{M(aE - J)^2}{r^3}$$

Dan

$$\frac{1}{2}\dot{r} + V_{eff} = \frac{1}{2}(E^2 - 1)$$

$$V_{eff} = \frac{1}{2}(E^2 - 1);\ \frac{dV_{eff}}{dr}|_{r=r_0} = 0$$

Dengan menggunakan $U = \frac{1}{r}$ dan disubtitusi ke persamaan ():

$$-MU + \frac{1}{2}U^2[J^2 - a^2(E^2 - 1)] - M(aE - J)^2U^3 = \frac{1}{2}(E^2 - 1) \qquad \text{(C.1)}$$

Diferensialkan terhadap U:

$$\frac{d}{dU}\left(-MU + \frac{1}{2}U^2[J^2 - a^2(E - 1)] - M(aE - J)^2U^3\right) = \frac{d}{dU}\left(\frac{1}{2}(E^2 - 1)\right)$$

$$-M + U[J^2 - a^2(E^2 - 1)] - 3M(aE - J)^2U^2 = 0$$

Kemudian masukkan variabel baru $x = (aE - J) \Longrightarrow J^2 = x^2 + a^2E^2 + 2axE$, maka persamaan (C.1):

$$-MU + \frac{1}{2}U^2\left[(x^2 + a^2E^2 + 2axE) - a^2(E^2 - 1)\right] - M(aE - J)^2U^3 = \frac{1}{2}(E^2 - 1)$$

$$-MU + \frac{1}{2}U^2(x^2 + a^2E^2 + 2axE - a^2E^2 + a^2) - Mx^2U^3 = \frac{1}{2}(E^2 - 1)$$

$$-MU + \frac{1}{2}(x^2 + 2axE + a^2)U^2 - Mx^2U^3 = \frac{1}{2}(E^2 - 1) \qquad \text{(C.2)}$$

Turunkan persamaan (C.2) terhadap U:

$$\frac{d}{du}\left(-MU + \frac{1}{2}U^2(x^2 + a^2E^2 + 2axE - a^2E^2 + a^2) - Mx^2U^3\right) = \frac{d}{du}\left(\frac{1}{2}(E^2 - 1)\right)$$

$$-M + (x^2 + +2axE + a^2)U - 3Mx^2U^2 = 0 \qquad \text{(C.3)}$$

Pers (C.2) masing-masing dikalikan dengan 2:

$$-MU + \frac{1}{2}(x^2 + 2axE + a^2)U^2 - Mx^2U^3 = \frac{1}{2}(E^2 - 1)$$

$$-2MU + (x^2 + 2axE + a^2)U^2 - 2Mx^2U^3 + 1 = E^2 \qquad \text{(C.4)}$$

Pers (C.3) dikalikan dengan U:

$$-MU + (x^2 + +2axE + a^2)U^2 - 3Mx^2U^3 = 0$$

$$(x^2 + +2axE + a^2)U^2 = MU + 3Mx^2U^3 \qquad \text{(C.5)}$$

Subtitusi (C.4) ke (C.5):

$$-2MU + MU + 3Mx^2U^3 - 2Mx^2U^3 + 1 = E^2$$

$$-MU + Mx^2U^3 + 1 = E^2$$

$$E^2 = (1 - MU) + Mx^2U^3 \tag{C.6}$$

Persamaan (C.6) disubtitusi ke pers (C.4)

$$-2MU + (x^2 + 2axE + a^2)U^2 - 2Mx^2U^3 + 1 = (1 - MU) + Mx^2U^3$$

$$-2MU + (x^2 + 2axE + a^2)U^2 - 2Mx^2U^3 + 1 - 1 + MU - Mx^2U^3 = 0$$

$$-MU + (x^2 + 2axE + a^2)U^2 - 3Mx^2U^3 = 0$$

$$-M + (x^2 + 2axE + a^2)U - 3Mx^2U^2 = 0$$

$$-M + x^2U + 2axEU + a^2U - 3Mx^2U^2 = 0$$

$$2axEU = 3Mx^2U^2 + M - x^2U - a^2U$$

$$2axEU = x^2U(3MU - 1) - (a^2 - M) \tag{C.7}$$

Dengan mengeliminasi E di antara persamaan (C.6) dan (C.7) didapatkan:

$$4a^2x^2U^2[(1 - MU) + Mx^2U^3] = x^4U^2(3MU - 1)^2 + (a^2U - M)^2 \\ -2x^2U(3MU - 1)(a^2U - M)$$

$$4a^2x^2U^2(1 - MU) + 4Ma^2x^4U^5 = x^4U^2(3MU - 1)^2 + (a^2U - M)^2 \\ -2x^2U(3MU - 1)(a^2U - M)$$

$$x^4[(3MU - 1)^2 - 4Ma^2U^3]U^2 + x^2[4a^2U(1 - MU) - 2(3MU - 1)(a^2U - \\ M)]U + (a^2U - M)^2 = 0$$

atau

$$U^2[(3MU - 1)^2 - 4Ma^2U^3]x^4 + 2U[2a^2U(MU - 1) - (3MU - 1)(a^2U - \\ M)]x^2 + (a^2U - M)^2 = 0$$

atau misalkan $Z = x^2$

$$U^2[(3MU - 1)^2 - 4Ma^2U^3]Z^2 \ + \\ 2U[2a^2U(MU - 1) - (3MU - 1)(a^2U - M)]Z + (a^2U - M)^2 = 0 \tag{C.8}$$

Gunakan rumus kuadrat:

$$Z_{12} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{C.9}$$

Pertama hitung nilai A, B, dan C:

$A = U^2[(3MU - 1)^2 - 4Ma^2U^3]$

$= U^2(9M^2U^2 - 6MU + 1 - 4Ma^2U^3)$

$B = 2U[2a^2U(MU - 1) - (3MU - 1)(a^2U - M)]$

$= 2U(2a^2MU^2 - 2a^2U - 3a^2MU^2 + 3M^2U + a^2U - M)$

$= 2U(3M^2U - a^2MU^2 - a^2U - M)$

$= 6M^2U^2 - 2a^2MU^3 - 2a^2U^2 - 2MU$

$C = (a^2U - M)^2$

$= (a^2U - M)(a^2U - M)$

$= a^4U^2 - a^2MU - a^2MU + M$

$= a^4U^2 - 2a^2MU + M$

Subtitusi nilai A, B, dan C ke dalam rumus kuadrat, didapatkan:

$$Z_{12} = \frac{\left(a\sqrt{U} \pm \sqrt{M}\right)^2}{U(1 - 3MU \mp 2a\sqrt{MU^3})} \qquad \text{(C.10)}$$

Mengingat $Z = x^2$, maka Jarak radial x dalam koordinat Boyer-Lindquist dinyatakan sebagai:

$$x^2 = \frac{\left(a\sqrt{U} \pm \sqrt{M}\right)^2}{U(1 - 3MU \mp 2a\sqrt{MU^3})}$$

$$x = \frac{\sqrt{\left(a\sqrt{U} \pm \sqrt{M}\right)^2}}{\sqrt{U(1 - 3MU \mp 2a\sqrt{MU^3})U}} = -\frac{a\sqrt{U} \pm \sqrt{M}}{[U(1 - 3MU \mp 2a\sqrt{MU^3})]^{\frac{1}{2}}} \qquad \text{(C.11)}$$

Kemudian solusi di atas disubtitusi ke dalam persamaan (39)

$$E^2 = (1 - MU) + Mx^2U^3$$

$$= (1 - MU) + M\left(-\frac{\left(a\sqrt{U} \pm \sqrt{M}\right)^2}{U\left(1 - 3MU \mp 2a\sqrt{MU^3}\right)}\right)U^3$$

$$= (1 - MU) + M\left(-\frac{a^2U + M \pm 2a\sqrt{UM}}{U\left(1 - 3MU \mp 2a\sqrt{MU^3}\right)}\right)U^3$$

$$= \frac{(1 - MU)\left(1 - 3MU \mp 2a\sqrt{MU^3}\right) + MU^2\left(a^2U + M \pm 2a\sqrt{UM}\right)}{\left(1 - 3MU \mp 2a\sqrt{MU^3}\right)}$$

$$= \frac{1-3MU\mp 2a\sqrt{MU^3}-MU+3M^2U^2\pm 2aMU\sqrt{MU^3}+MU^2a^2+M^2U^2\pm 2aMU\sqrt{U^3M}}{1-3MU\mp 2a\sqrt{MU^3}}$$

Suku pada pembilang dapat disederhanakan:

$$1-4MU\mp 2a\sqrt{MU^3}+4M^2U^2\pm 4aMU\sqrt{MU^3}+MU^3a^2$$

Karena $(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$ maka diperoleh:

$$= \left(-1+2MU\pm a\sqrt{MU^3}\right)^2$$

Kemudian masukkan kembali ke persamaan awal:

$$E^2 = \frac{\left(-1+2MU\pm a\sqrt{MU^3}\right)^2}{\left(1-3MU\mp 2a\sqrt{MU^3}\right)}$$

$$E = \frac{\left(-1+MU\pm a\sqrt{MU^3}\right)}{\left(1-3MU\mp 2a\sqrt{MU^3}\right)^{\frac{1}{2}}} \quad \text{(C.12)}$$

Selanjutnya untuk nilai J:

$$x = aE - J$$

$$J = x + aE$$

$$J = \left(-\frac{a\sqrt{U}\pm\sqrt{M}}{\left[U\left(1-3MU\mp 2a\sqrt{MU^3}\right)\right]^{\frac{1}{2}}}\right) + a\left(\frac{\left(-1+MU\pm a\sqrt{MU^3}\right)}{\left(1-3MU\mp 2a\sqrt{MU^3}\right)^{\frac{1}{2}}}\right)$$

$$J = \frac{-a\sqrt{U}\pm\sqrt{M}+a\sqrt{U}\left(-1+MU\pm a\sqrt{MU^3}\right)}{\sqrt{U}\left(1-3MU\mp 2a\sqrt{MU^3}\right)^{\frac{1}{2}}}$$

$$J = \mp\frac{\sqrt{M}\left(1+a^2U^2\pm 2a\sqrt{MU^3}\right)}{\sqrt{U}\left(1-3MU\mp 2a\sqrt{MU^3}\right)^{\frac{1}{2}}} \quad \text{(C.13)}$$

D. Persamaan ISCO

Dari persamaan (50):

$$U = \frac{J^2+a^2(E^2-1)}{6Mx^2}$$

Subtitusi nilai J^2, E^2 dan x^2 dari persamaan (44, 45 dan 46):
Untuk x_1, J_1, E_1, Misal $Q_- = 1-3MU-2a\sqrt{MU^3}$

$$J^2 = \frac{M\left(1+a^2U^2+2a\sqrt{MU^3}\right)^2}{U(1-3MU-2a\sqrt{MU^3}} = \frac{M\left(1+a^2U^2+2a\sqrt{MU^3}\right)^2}{UQ_-} \quad \text{(D.1)}$$

$$E^2 = \frac{\left(1-2MU+a\sqrt{MU^3}\right)^2}{1-3MU-2a\sqrt{MU^3}} = \frac{\left(1-2MU+a\sqrt{MU^3}\right)^2}{Q_-} \quad \text{(D.2)}$$

$$x^2 = \frac{\left(a\sqrt{U}+\sqrt{M}\right)^2}{U(1-3MU-2a\sqrt{MU^3})} = \frac{a^2U+M+2a\sqrt{MU}}{UQ_-} \quad \text{(D.3)}$$

$$U = \left[\frac{M\left(1+a^2U^2+2a\sqrt{MU^3}\right)^2}{UQ_-} + a^2\left(\frac{\left(1-2MU+a\sqrt{MU^3}\right)^2}{Q_-} - 1\right)\right] \times \frac{UQ_-}{6M(a^2U+M+2a\sqrt{MU})}$$

$$\begin{aligned}
6MU\left(a^2U + M + \sqrt{MU}\right) &= M\left(1 + a^2U^2 + 2a\sqrt{MU^3}\right)^2 - a^2U\left(1 - 2MU + a\sqrt{MU^3}\right)^2 + a^2U\left(1 - 3MU - 2a\sqrt{MU^3}\right) \\
&= M\left(1 + a^4U^4 + 4Ma^2U^3 + 2a^2U^2 + 4a\sqrt{MU^3} + 4a^3U^2\sqrt{MU^3}\right) - a^2U\left(1 + 4M^2U^2 + Ma^2U^3 - 4MU - 2a\sqrt{MU^3} + 4MaU\sqrt{MU^3}\right) + a^2U - 3Ma^2U^2 - 2a^3U\sqrt{MU^3} \\
&= M + Ma^4U^4 + 4M^2a^2U^2 + 2Ma^2U^2 + 4Ma\sqrt{MU^3} + 4Ma^3U^2\sqrt{MU^3} - a^2U - 4M^2a^2U^3 - Ma^4U^4 + 4Ma^2U^2 + 2a^3U\sqrt{MU^3} - 4Ma^2U^2\sqrt{MU^3} + a^2U - 3Ma^2U^2 - 2a^3U\sqrt{MU^3} \\
&= M + 3Ma^2U^2 + 4Ma\sqrt{MU^3} + 8Ma^3U^2\sqrt{MU^3} - 2a^3U\sqrt{MU^3}
\end{aligned}$$

$$3Ma^2U^3 + 6M^2U + 8MaU\sqrt{MU} - M = 0$$

atau

$$3a^2U^2 + 6MU + 8a\sqrt{MU^3} - 1 = 0$$

Masukkan $U = \frac{1}{r}$:

$$r^2 - 6Mr - 3a^2 - 8a\sqrt{Mr} = 0 \quad \text{(D.4)}$$

Dengan cara yang sama untuk x_2, J_2, E_2 misal $Q_+ = 1 - 3MU + 2a\sqrt{MU^3}$

$$x^2 = \frac{\left(a\sqrt{U}-\sqrt{M}\right)^2}{U(1-3MU+2a\sqrt{MU^3})} = \frac{a^2U+M-2a\sqrt{MU}}{UQ_+} \quad \text{(D.5)}$$

$$E^2 = \frac{\left(1-2MU+a\sqrt{MU^3}\right)^2}{1-3MU+2a\sqrt{MU^3}} = \frac{\left(1-2MU+a\sqrt{MU^3}\right)^2}{Q_+} \quad \text{(D.6)}$$

$$J^2 = \frac{M\left(1+a^2U^2-2a\sqrt{MU^3}\right)^2}{U(1-3MU+2a\sqrt{MU^3}} = \frac{M\left(1+a^2U^2-2a\sqrt{MU^3}\right)^2}{UQ_+} \quad \text{(D.7)}$$

Maka:

$$r^2 - 6Mr - 3a^2 + 8a\sqrt{Mr} = 0 \quad \text{(D.8)}$$

Dengan demikian persamaan ISCO didapatkan:

$$r^2 - 6Mr - 3a^2 \pm 8a\sqrt{Mr} = 0$$