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Research Article

Clifford-Valued Shearlet Transforms on $\mathrm{Cl}_{(P,Q)}$-Algebras

Firdous A. Shah[1], **Aajaz A. Teali**[1], **and Mawardi Bahri**[2]

[1]*Department of Mathematics, University of Kashmir, South Campus, Anantnag 192101, Jammu and Kashmir, India*
[2]*Department of Mathematics, Hasanuddin University, Makassar, Indonesia*

Correspondence should be addressed to Mawardi Bahri; mawardibahri@gmail.com

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The shearlet transform is a promising and powerful time-frequency tool for analyzing nonstationary signals. In this article, we introduce a novel integral transform coined as the Clifford-valued shearlet transform on $\mathrm{Cl}(p,q)$ algebras which is designed to represent Clifford-valued signals at different scales, locations, and orientations. We investigated the fundamental properties of the Clifford-valued shearlet transform including Parseval's formula, isometry, inversion formula, and characterization of range using the machinery of Clifford Fourier transforms. Moreover, we derived the pointwise convergence and homogeneous approximation properties for the proposed transform. We culminated our investigation by deriving several uncertainty principles such as the Heisenberg–Pauli–Weyl uncertainty inequality, Pitt's inequality, and logarithmic and local-type uncertainty inequalities for the Clifford-valued shearlet transform.

1. Introduction

Wavelet transforms have been proved to be a successful tool for analyzing nontransient signals and have been applied in a number of fields including signal and image processing, differential and integral equations, sampling theory, quantum mechanics, medicine, and so on [1]. However, the efficiency of the wavelet transforms is considerably reduced when applied to higher dimensional signals as they are not able to capture the geometric features like edges and corners at different scales efficiently. The detection of such geometric features in nontransient signals is often highly desirable in numerous practical applications such as medical imaging, remote sensing, crystallography, and several other areas. To circumvent these constraints, a number of novel directional representation systems have been introduced and employed in recent years, such as the wedgelets, ridgelets, ripplets, curvelets, contourlets, surfacelets, brushlets, and shearlets. Among all these geometrical wavelet systems, the shearlet systems have been widely acknowledged and emerged as one of the most effective frameworks for representing multidimensional data because they are nonisotropic nature, and they offer optimally sparse representations [2], allow compactly supported analysing elements [3], are associated with fast decomposition and reconstruction algorithms, and provide a unified treatment of continuum and digital data [4, 5].

Clifford algebras have dethroned both the Grossmann's exterior algebra and Hamilton's quaternion algebra in the sense that they incorporate both the geometrical and algebraic features of Euclidean space into a single structure [6]. As a result, the theory of Clifford algebras has attained an overwhelming response and gained a respectable status in higher-dimensional signal and image processing mainly due to the reason that such algebras encompass all dimensions at once unlike the multidimensional tensorial approach with tensor products of one-dimensional phenomena. This true multidimensional nature allows specific constructions of higher dimensional signal and image processing tools including the Clifford Fourier transforms [7, 8], Clifford Gabor transforms, Clifford wavelet transforms, and other integral transforms in general [9–13].

Motivated and inspired by the contemporary developments in the theory of shearlet transforms abreast the profound applicability of the Clifford algebras, we introduce the notion of Clifford-valued shearlet transforms on $Cl_{(p,q)}$

algebras in the context of multidimensional signal analysis. Unlike the conventional shearlet transform, the proposed transform inherits both the geometric and algebraic properties of shearlet transforms and Clifford algebras. Although a meek analgoue of shearlet transform in the Clifford domain has been proposed in [14], it only deals with the $Cl_{(0,n)}$algebra, where $n = 3\text{mod}4$. Therefore, the centre piece of this study is to construct the Clifford-valued shearlets and the corresponding shearlet transforms in the most general setting $Cl_{(p,q)}$ by employing translations, sharing, scaling, and spinning elements. Besides, we study the basic properties of the Clifford-valued shearlet transforms including Parseval's and inversion formulae and range theorem using the machinery of Clifford Fourier transforms. Moreover, we derive the pointwise convergence and homogeneous approximation properties for the proposed transform. Finally, we formulate some uncertainty inequalities including the classical Heisenberg–Pauli–Weyl inequality, Pitt's inequality, and logarithmic inequality for the Clifford shearlet transforms.

The structure of this article is as follows. Section 2 deals with the preliminaries of Clifford algebras, whereas a comprehensive analysis of the general Clifford-valued shearlet transforms is carried out in Section 3. In Section 4, we study the homogeneous approximation properties for proposed transform. Several uncertainty principles for the proposed transform are also being studied in Section 5. Finally, a conclusion is summarized in Section 6.

2. Basics of Clifford Algebras

In this section, we present a brief overview of the Clifford algebras including the definitions of Clifford Fourier transforms, spin group, and some unitary operators.

The Clifford algebra $Cl_{(p,q)}$ is a noncommutative, associative algebra generated by the orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the n-dimensional Euclidean space $\mathbb{R}^n$ governed by the multiplication rule:

$$e_i e_j + e_j e_i = 2\varepsilon_i \delta_{ij}, \quad i, j = 1, 2, \ldots, n, \tag{1}$$

where $n = p + q$, $\varepsilon_i = +1$ for $i = 1, 2, \ldots, p$ and $\varepsilon_i = -1$ for $i = p+1, p+2, \ldots, n$, with δ_{ij} denoting the usual Kronecker's delta function. The noncommutative product and the additional axiom of associativity generates the 2^n- dimensional Clifford geometric algebra $Cl_{(p,q)}$, which can be decomposed as

$$Cl_{(p,q)} = \bigoplus_{k=0}^{n} Cl^k_{(p,q)}, \tag{2}$$

where $Cl^k_{(p,q)}$ denotes the space of k-vectors given by

$$Cl^k_{(p,q)} := \text{span}\left\{e_{i_1}, e_{i_2}, \ldots, e_{i_k}; i_1 \le i_2 \le \ldots \le i_k\right\}. \tag{3}$$

Any general element of the Clifford algebra is called a multivector and every multivector $M \in Cl_{(p,q)}$ can be represented in the following form:

$$M = \sum_A M_A e_A = \langle M\rangle_0 + \langle M\rangle_1 + \cdots + \langle M\rangle_n, \quad M_A \in \mathbb{R}, A \subset \{1, 2, \ldots, n\}, \tag{4}$$

where $e_A = e_{i_1} e_{i_2} \ldots e_{i_k}$ and $i_1 \le i_2 \le \ldots \le i_k$. Moreover, $\langle\cdot\rangle_k$ is called as the grade k-part of M, and $\langle\cdot\rangle_0, \langle\cdot\rangle_1, \langle\cdot\rangle_2, \ldots$, respectively, denote the scalar part, vector part, bivector part, and so on. The Clifford conjugate of a multivector $M \in Cl_{(p,q)}$ is given by

$$\overline{M} = \sum_{r=0}^{n} (-1)^{r(r-1)/2} \overline{\langle M\rangle_r}, \tag{5}$$

where the scalar product of multivectors M and $\overline{N}$ is defined as

$$Sc(M\overline{N}) = |MN| = M \star \overline{N} = \sum_A M_A N_A. \tag{6}$$

Moreover, for any pair of multivectors $M, N \in Cl_{(p,q)}$, it can be easily verified that

$$|MN| \le 2^n |M||N|. \tag{7}$$

We now intend to recall the fundamental notion of Clifford Fourier transforms in $L^r(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$, $1 \le r < \infty$ as

$$L^r\left(\mathbb{R}^{(p,q)}, Cl_{(p,q)}\right) = \left\{\mathbf{f}\colon \mathbb{R}^{(p,q)} \longrightarrow Cl_{(p,q)}; \|\mathbf{f}\|_r = \left(\int_{\mathbb{R}^{(p,q)}} |\mathbf{f}(x)|^r d^n x\right)^{1/r} < \infty\right\}. \tag{8}$$

It is imperative to mention that any function $\mathbf{f} \in L^r(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$ can be expressed as a combination of the real-valued functions f_A and the basis elements e_A as

$$\mathbf{f}(x) = \sum_A f_A(x) e_A. \tag{9}$$

Due to the noncommutativity of Clifford-valued functions, several analogues of the Clifford Fourier transforms have been introduced in the literature. However, we shall be interested in following definition due to Bahri et al. [15].

Definition 1. Let $I \in Cl_{(p,q)}$ be a square root of -1. The Clifford Fourier transform of any function $\mathbf{f} \in L^1(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$ is defined by

$$\mathcal{F}_{Cl}[f(x)](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^{(p,q)}} f(x) e^{-Iv(\xi,x)} \mathrm{d}^n x, \qquad (10)$$

where $nx, \xi \in \mathbb{R}^{(p,q)}$, $\mathrm{d}^n x = \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n$, $v: \mathbb{R}^{(p,q)} \times \mathbb{R}^{(p,q)} \longrightarrow \mathbb{R}$, and $v(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$.

The inversion and Plancherel formulae associated with the Clifford Fourier transform (10) are given by

$$\mathbf{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^{(p,q)}} \mathcal{F}_{Cl}[\mathbf{f}(x)](\xi) e^{Iv(\xi,x)} \mathrm{d}^n \xi,$$
$$\langle \mathbf{f}, \mathbf{g} \rangle_{L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})} = \langle \mathcal{F}_{Cl}[\mathbf{f}], \mathcal{F}_{Cl}[\mathbf{g}] \rangle_{L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})}, \qquad (11)$$

In this case, the inner product of two multivector functions $\mathbf{f}$ and $\mathbf{g}$ is described through

$$\langle \mathbf{f}, \mathbf{g} \rangle_{L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})} = \int_{\mathbb{R}^{(p,q)}} \mathbf{f}(x) \overline{\mathbf{g}(x)} \mathrm{d}^n x, \qquad (12)$$

and its scalar part is given by

$$\begin{aligned} |\langle \mathbf{f}, \mathbf{g} \rangle|_{L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})} &= \int_{\mathbb{R}^{(p,q)}} |\mathbf{f}(x) \mathbf{g}(x)|^2 \mathrm{d}^n x \\ &= \int_{\mathbb{R}^{(p,q)}} Sc(\mathbf{f}(x) \overline{\mathbf{g}(x)}) \mathrm{d}^n x \\ &= Sc\left(\int_{\mathbb{R}^{(p,q)}} f(x) \overline{g(x)} \mathrm{d}^n x \right). \end{aligned} \qquad (13)$$

For an efficient representation of Clifford-valued functions, we employ the spin elements obtained from the spin group as defined below.

Definition 2. The spin-group is a double covering of special orthogonal group of $\mathbb{R}^n$ and is defined by

$$\mathrm{Spin}(n) = \left\{ \mathbf{r} \in Cl^+_{(p,q)}; \bar{\mathbf{r}}\mathbf{r} = \mathbf{r}\bar{\mathbf{r}} = 1 \right\}, \qquad (14)$$

where $Cl^+_{(p,q)}$ is a subgroup of the invertible elements in the Clifford algebra $Cl_{(p,q)}$.

To facilitate the construction of Clifford-valued shearlets, we define the fundamental unitary operators acting on the space $L^r(\mathbb{R}^{(p,q)})$. For $a > 0$, $s \in \mathbb{R}^{n-1}$ and $b \in \mathbb{R}^n$, and the scaling, shearing, spin-rotation, and translation operators are denoted by D_{A_a}, $\mathcal{D}_{S_s}$, $\mathcal{R}_r$, T_b, respectively, and are defined as

$$\begin{aligned} D_{A_a} \Psi(x) &= \left| \det A_a \right|^{-1/2} \Psi\left(A_a^{-1} x \right), \\ \mathcal{D}_{S_s} \Psi(x) &= \Psi\left(S_s^{-1} x \right), \\ \mathcal{R}_r \Psi(x) &= \mathbf{r} \Psi(\bar{\mathbf{r}} x \mathbf{r}) \bar{\mathbf{r}}, \\ T_b \Psi(x) &= \Psi(x - b), \end{aligned} \qquad (15)$$

and the matrices involved in equation (15) are

$$\begin{aligned} A_a &= \begin{pmatrix} a & \mathbf{0}^T_{n-1} \\ \mathbf{0}_{n-1} & \mathrm{sgn}(a)|a|^{1/n} I_{n-1} \end{pmatrix}, \\ S_s &= \begin{pmatrix} 1 & \mathbf{s}^T \\ \mathbf{0}_{n-1} & I_{n-1} \end{pmatrix}, \end{aligned} \qquad (16)$$

where $\mathbf{s}^T = (s_1, s_2, \ldots, s_{n-1})$, and $\mathrm{sgn}(\cdot)$ and $\mathbf{0}$ denote the well-known Signum function and the null vector, respectively. Moreover, the composition of the scaling matrix A_a and the shearing matrix S_s is given by

$$S_s A_a = \begin{pmatrix} a & \mathrm{sgn}(a) a^{1/n} s_1 & \mathrm{sgn}(a) a^{1/n} s_2 & \mathrm{sgn}(a) a^{1/n} s_3 & \cdots & \mathrm{sgn}(a) a^{1/n} s_{n-1} \\ 0 & \mathrm{sgn}(a) a^{1/n} & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \mathrm{sgn}(a) a^{1/n} \end{pmatrix}. \qquad (17)$$

3. The Clifford-Valued Shearlet Transform on $Cl_{(p,q)}$ Algebras

In this section, we shall construct the Clifford-valued shearlets on $Cl_{(p,q)}$ algebras by using the combined action of the scaling, sharing, spin-rotation and translation operators. Besides, we study the fundamental properties of the Clifford-valued shearlet transform including Parseval's formula, inversion formula, and obtain a complete characterization of the range. Prior to that, we shall demonstrate that the novel family of Clifford-valued shearlets is endowed with an affine group structure.

Consider that the set $\mathcal{G} = \mathbb{R}^+ \times \mathrm{Spin}(n) \times \mathbb{R}^{n-1} \times \mathbb{R}^n$ endowed with the binary operation $\odot$ is defined as

$$\begin{aligned} &(a, \mathbf{r}, s, b) \odot (a', \mathbf{r}', s', b') \\ &\quad = \left(aa', \mathbf{r} + \mathbf{r}', s + a^{1-(1/n)} s', b + S_s A_a b' \right), \end{aligned} \qquad (18)$$

where $a, a' \in \mathbb{R}^+, s, s' \in \mathbb{R}^{n-1}, b, b' \in \mathbb{R}^n, r, r' \in \mathrm{Spin}(n)$. Clearly, $(1, \mathbf{0}, 0_{n-1}, 0_n,)$ is the neutral element of $\mathcal{G}$, whereas $(a^{-1}, -\mathbf{r}, -a^{1/n-1} s, -A_a^{-1} S_s^{-1} b)$ is the inverse of any arbitrary element $(a, \mathbf{r}, s, b) \in \mathcal{G}$. Moreover, it is easy to verify that

$$((a,\mathbf{r},s,b)\odot(a',\mathbf{r}',s',b'))\odot(a'',\mathbf{r}'',s'',b'')$$
$$=(a,\mathbf{r},s,b)\odot((a',\mathbf{r}',s',b')\odot(a'',\mathbf{r}'',s'',b'')). \tag{19}$$

Hence, we conclude that $(\mathscr{G},\odot)$ constitutes a group and is formally called as the similitude group of dilations, translations, shearing, and spinning.

Furthermore, we claim that the left Haar measure on $\mathscr{G}$ is given by $\mathrm{d}\eta=\mathrm{d}a\mathrm{d}r\mathrm{d}^{n-1}s\mathrm{d}^n b/a^{n+1}$. In fact, for any function $\mathbf{f}\in L^2(\mathscr{G},Cl_{(p,q)})$, we have

$$\int_{\mathscr{G}}\mathbf{f}[(a,\mathbf{r},s,b)\odot(a',\mathbf{r}',s',b')]\mathrm{d}\eta=\int_{\mathbb{R}^+\times Spin(n)\times\mathbb{R}^n\times\mathbb{R}^n} f\left[\left(aa',\mathbf{r}+\mathbf{r}',s+a^{1-(1/n)}s',b+S_sA_ab'\right)\right]\frac{\mathrm{d}a\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b}{a^{n+1}}. \tag{20}$$

Making use of the substitution $\tilde{a}:=aa',\tilde{\mathbf{r}}:=\mathbf{r}\mathbf{r}',\tilde{s}:=s+(a)^{1-1/n}s',\tilde{b}:=b+S_sA_ab'$, i.e., $\mathrm{d}a=(a')^{-1}\mathrm{d}\tilde{a}$, $\mathrm{d}\mathbf{r}=\mathrm{d}\tilde{\mathbf{r}},\mathrm{d}^{n-1}s=(a)^{-((n-1)^2/n)}\mathrm{d}^{n-1}\tilde{s},\mathrm{d}^n b=(a)^{-2+1/n}\mathrm{d}^n\tilde{b}$, the above expression becomes

$$\begin{aligned}\int_{\mathscr{G}} f[(a,r,s,b)\odot(a',r',s',b')]\mathrm{d}\eta&=\int_{\mathbb{R}^+\times\mathbb{R}^n\times\mathrm{Spin}(n)\times\mathbb{R}^n} f[(\tilde{a},\tilde{r},\tilde{s},\tilde{b})]\frac{(a')^{-1}\mathrm{d}\tilde{a}\mathrm{d}\tilde{r}(a)^{-((n-1)^2/n)}\mathrm{d}^{n-1}\tilde{s}(a)^{-2+1/n}\mathrm{d}^n\tilde{b}}{(\tilde{a}/a')^{n+1}}\\&=\int_{\mathbb{R}^+\times\mathbb{R}^n\times\mathrm{Spin}(n)\times\mathbb{R}^n} f[(\tilde{a},\tilde{r},\tilde{s},\tilde{b})]\frac{\mathrm{d}\tilde{a}\mathrm{d}\tilde{r}\mathrm{d}^{n-1}\tilde{s}\mathrm{d}^n\tilde{b}}{(\tilde{a})^{n+1}},\end{aligned} \tag{21}$$

which validates the claim that $\mathrm{d}\eta=\mathrm{d}a\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b/a^{n+1}$ is indeed the left Haar measure on $\mathscr{G}$.

Next, we shall construct a novel class of shearlet systems on $Cl_{(p,q)}$ algebras by the combined action of the scaling D_{A_a}, sharing $\mathscr{D}_{S_s}$, spin-rotation $\mathscr{R}_{\mathbf{r}}$, and translation T_b operators on any analyzing function $\Psi\in L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})$.

For any $a\in\mathbb{R}^+,s\in\mathbb{R}^{n-1},b\in\mathbb{R}^n$, and $\mathbf{r}\in\mathrm{Spin}(n)$, consider the family of analyzing functions:

$$\Psi^r_{a,s,b}(x)=\left\{D_{A_a}\mathscr{D}_{S_s}\mathscr{R}_r T_b\Psi(x)=a^{(1/2n)-1}\mathbf{r}\Psi\left(A_a^{-1}S_s^{-1}\bar{\mathbf{r}}(x-b)\mathbf{r}\right)\bar{\mathbf{r}}\right\}, \tag{22}$$

which is called as the family of Clifford-valued shearlets on the geometric algebra-$C\ell_{(p,q)}$. The system of functions (22) satisfies the following properties:

(i) The system (22) is a dense subspace of $L^2(\mathbb{R}^n,Cl_{(p,q)})$

(ii) The following norm equality holds good:

$$\left\|\Psi^r_{a,s,b}\right\|_{L^2\left(\mathbb{R}^n,Cl_{(p,q)}\right)}=\|\Psi\|_{L^2\left(\mathbb{R}^n,Cl_{(p,q)}\right)}. \tag{23}$$

(iii) The Clifford Fourier transform of the family of functions $\Psi^{\mathbf{r}}_{a,s,b}(x)$ reads

$$\mathscr{F}_{Cl}\left[\Psi^{\mathbf{r}}_{a,s,b}\right](\xi)=a^{1-(1/2n)}\mathscr{F}_{Cl}[r\Psi(\cdot)\bar{\mathbf{r}}]\left(\mathbf{r}S_sA_a\xi\bar{\mathbf{r}}\right)e^{-Iv(\xi,b)}. \tag{24}$$

Next, we shall present the notion of an admissible Clifford-valued shearlet on the space of Clifford-valued functions $L^2(\mathbb{R}^n,Cl_{(p,q)})$.

Definition 3 (Admissibility). A nontrivial function $\Psi\in L^2(\mathbb{R}^n,Cl_{(p,q)})$ is called an admissible Clifford-valued shearlet if

$$C_\Psi=\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\overline{\left\{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\bar{\mathbf{r}}]\left(\mathbf{r}S_sA_a\xi\bar{\mathbf{r}}\right)\right\}}\left\{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\bar{\mathbf{r}}]\left(\mathbf{r}S_sA_a\xi\bar{\mathbf{r}}\right)\right\}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}\mathbf{r}}{a^{(n^2-n+1/n)}}, \tag{25}$$

which is an invertible multivector and finite, i.e., $\xi\in\mathbb{R}^{(p,q)}$.

Remark 1. It is worth noticing that $\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\bar{\mathbf{r}}](0)=0$, for $\xi=0$; that is, $\Psi(x)=\sum_A\Psi_A(x)e_A$. and

$$\int_{\mathbb{R}^{(p,q)}}\Psi_A(x)e_Ae^{-Iv(0,x)}\mathrm{d}^n x=0, \tag{26}$$

which in turn implies that for every component Ψ_A of the Clifford-valued shearlet Ψ is zero; that is,

$$\int_{\mathbb{R}^{(p,q)}}\Psi_A(x)\mathrm{d}^n x=0. \tag{27}$$

Based on the novel family of Clifford-valued shearlets defined in equation (22), we have the following main definition of the continuous Clifford-valued shearlet transform.

Definition 4. The continuous Clifford-valued shearlet transform of any multivector signal $\mathbf{f} \in L^2(\mathbb{R}^n, Cl_{(p,q)})$ with respect to an analysing Clifford-valued shearlet $\Psi \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$ is defined by

$$\begin{aligned}\mathscr{C}\mathscr{S}_\Psi \mathbf{f}(a,r,s,b) &= \langle f, \Psi^r_{a,s,b}\rangle_{L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})} \\ &= \int_{\mathbb{R}^{(p,q)}} \mathbf{f}(x)\overline{\Psi^r_{a,s,b}(x)}\mathrm{d}^n x.\end{aligned} \tag{28}$$

where $\Psi^{\mathbf{r}}_{a,s,b}(x)$ is given by equation (22).

The corresponding spectral representation of the Clifford-valued shearlet transform is

$$\begin{aligned}\mathscr{C}\mathscr{S}_\Psi \mathbf{f}(a,\mathbf{r},s,b) = a^{1-(1/2n)} &\int_{\mathbb{R}^n} \mathscr{F}_{Cl}[\mathbf{f}](\xi) \\ &\cdot e^{Iv(\xi,b)}\overline{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_s A_a \xi\overline{\mathbf{r}})}\mathrm{d}^n\xi.\end{aligned} \tag{29}$$

We now present an example for the lucid illustration of the proposed Clifford-valued shearlet transform (28).

Example 1. Consider the Clifford-valued Hermite wavelets [16]as

$$\begin{aligned}\psi_n(x) &= (-1)^n \partial_x^n\left[\exp\left(-\frac{|x|^2}{2}\right)\right], \\ \partial_x^n &= \left(\frac{\partial^n}{\partial x_1} + \frac{\partial^n}{\partial x_2} + \cdots + \frac{\partial^n}{\partial x_n}\right).\end{aligned} \tag{30}$$

Therefore, the corresponding Clifford-valued shearlets of $\psi_n(x)$ are obtained as

$$\Psi^{\mathbf{r}}_{a,s,b}(x) = \left|\det A_a\right|^{-1/2}\mathbf{r}\left(A_a^{-1}S_s^{-1}(x-b)^n\right)\exp\left(-\frac{\left|A_a^{-1}S_s^{-1}(x-b)\right|^2}{2}\right)\overline{\mathbf{r}}, \tag{31}$$

and the Clifford-valued shearlet transform (28) of any function $\mathbf{f} \in L^2(\mathbb{R}^n, Cl_n)$, with respect to the analyzing shearlets (31) can be computed as

$$\begin{aligned}\mathscr{C}\mathscr{S}_\Psi \mathbf{f}(a,\mathbf{r},s,b) = \left|\det A_a\right|^{-1/2}&\int_{\mathbb{R}^n}\mathbf{f}(x)\mathbf{r}\left(A_a^{-1}S_s^{-1}(x-b)^n\right) \\ &\times\exp\left(-\frac{\left|A_a^{-1}S_s^{-1}(x-b)\right|^2}{2}\right)\overline{\mathbf{r}}\mathrm{d}^n x.\end{aligned} \tag{32}$$

For simplicity, we shall compute the two-dimensional Clifford-valued shearlet transform for the given function $\mathbf{f}$ with respect to the shearlets:

$$\begin{aligned}\Psi^{\mathbf{r}}_{a,s,b}(x_1,x_2) = \left|\det A_a\right|^{-1/2}&\left(A_a^{-1}S_s^{-1}\left[(x_1-b_1)^2,(x_2-b_2)^2\right]\right) \\ &\times\exp\left(-\frac{\left|A_a^{-1}S_s^{-1}(x_1-b_1,x_2-b_2)\right|^2}{2}\right),\end{aligned} \tag{33}$$

where $A_a = \begin{bmatrix} a & 0 \\ 0 & a^{1/2}\end{bmatrix}, S_s = \begin{bmatrix} 1 & s \\ 0 & 1\end{bmatrix}$. After simplifying, we obtain

$$\begin{aligned}\Psi^{\mathbf{r}}_{a,s,b}(x_1,x_2) = \sqrt{a}&\left((x_1-b_1)^2 - \left(s-a^{3/2}\right)(x_2-b_2)^2\right) \\ &\times\exp\left(-\frac{(x_1-b_1)^2 + \left(s-a^{-(1/2)}\right)(x_2-b_2)^2}{2a^2}\right).\end{aligned} \tag{34}$$

The two-dimensional analyzing shearlets $\psi^{\mathbf{r}}_{a,s,b}(x_1,x_2)$ given by equation (34) at different values of $a, s, \mathbf{r}$, and b are plotted in Figure 1. The parameters a and s determine the scaling anisotropy and the decaying rate of shearlets providing more accurate location and orientation. In comparison with wavelets, shearlets not only inherits advantages of wavelets $(s = 0)$ but also provide detailed information of position, normal and curvature of discontinuities.

The Clifford-valued shearlet transform of $\mathbf{f}(x_1,x_2) = \exp\{-(x_1^2+x_2^2)/2\}$ is computed as

$$\mathscr{C}\mathscr{S}_\psi \mathbf{f}(a,\mathbf{r},s,b) = \sqrt{a}\int_{\mathbb{R}^2}\left((x_1-b_1)^2 - \left(s-a^{3/2}\right)(x_2-b_2)^2\right)\times\exp\left(-\frac{(x_1-b_1)^2 + \left(s-a^{-1/2}\right)(x_2-b_2)^2 + a^2\left(x_1^2+x_2^2\right)}{2a^2}\right)\mathrm{d}x_1\mathrm{d}x_2. \tag{35}$$

For different values of $a, s, \mathbf{r}$, and b, the corresponding Clifford-valued shearlet transforms of $\mathbf{f}(x_1,x_2)$ with respect the analysing shearlets (34) are depicted in Figure 2 after computing the integrals (35) in *Mathematica* software. From the simulation, we infer that the Clifford-valued shearlet transform enables a precise characterization of location, orientation, and curvature of discontinuities in two dimensional signals.

In the following theorem, we assemble some of the basic properties of the Clifford-valued shearlet transform (28).

Theorem 1. *$\Psi^{\mathbf{r}}_{a,s,b}(x_1,x_2)$for $\mathbf{f},\mathbf{g} \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$, and admissible Clifford-valued shearlets Ψ and Φ. The continuous Clifford-valued shearlet transform (28) satisfies the following properties:*

(i) *Linearity*: $\mathscr{C}\mathscr{S}_\Psi(\alpha\mathbf{f}+\beta\mathbf{g})(a,\mathbf{r},s,b) = \alpha\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b) + \beta\mathscr{C}\mathscr{S}_\Psi\mathbf{g}(a,\mathbf{r},s,b), \alpha,\beta \in \mathbf{Cl}_{(\mathbf{p},\mathbf{q})}$

(ii) *Anti – linearity*: $\mathscr{C}\mathscr{S}_{\alpha\Psi+\beta\Phi}\mathbf{f}(a,\mathbf{r},s,b) = \mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\overline{\alpha} + \mathscr{C}\mathscr{S}_\Phi\mathbf{f}(a,\mathbf{r},s,b)\overline{\beta}$

(iii) *Translation covariance*: $\mathscr{C}\mathscr{S}_\Psi(T_k\mathbf{f})(a,\mathbf{r},s,b) = \mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b-k)$

(iv) *Dilation covariance*: $\mathscr{C}\mathscr{S}_\Psi(\mathscr{D}_{(1/\gamma)}\mathbf{f}(x))(a,\mathbf{r},s,b) = (\mathscr{C}\mathscr{S}_{\mathscr{D}_\gamma\Psi}\mathbf{f}(x))(a,\mathbf{r},s,(b/\gamma)), \gamma \in \mathbb{R}$

(v) *Parity*: $\mathscr{C}\mathscr{S}_\Psi(P\mathbf{f}(x))(a,\mathbf{r},s,b) = (-1)^n\mathscr{C}\mathscr{S}_{P\Psi}(\mathbf{f}(x))(a,\mathbf{r},s,-b), P\mathbf{f}(x) = \mathbf{f}(-x)$

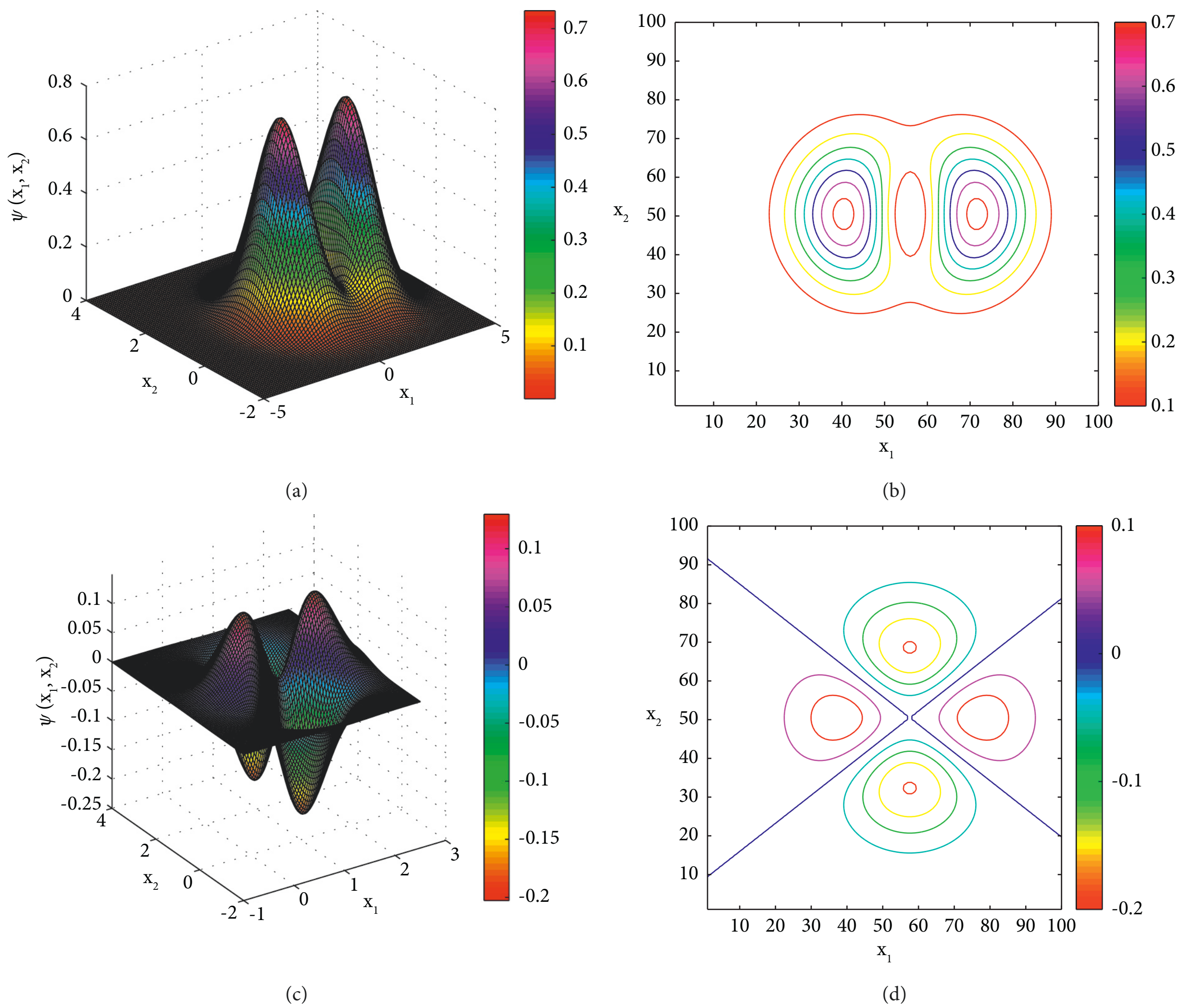

FIGURE 1: Two-dimensional analyzing shearlets $\psi^{\mathbf{r}}_{a,s,b}(x_1, x_2)$ given by equation (34) at different values of $a, \mathbf{r}, b$, and s. (a) 2D-shearlets at $a = 1$, $b = 1$, and $s = 0$. (b) Contour plot of 2D-shearlets at $a = 1$, $b = 1$, and $s = 0$. (c) 2D-shearlets at $a = 1/2$, $b = 1$ and $s = 1$. (d) Contour plot of 2D-shearlets at $a = 1/2$, $b = 1$ and $s = 1$.

(vi) *Translation in* Ψ: $\mathscr{C}\mathscr{S}_{T_k\Psi}(\mathbf{f}(x))(a, \mathbf{r}, s, b) = \mathscr{C}\mathscr{S}_{\Psi}(\mathbf{f}(x))(a, \mathbf{r}, s, b + k)$

Proof. For the sake of brevity, we omit the proof.

In our next theorem, we show that the Clifford-valued shearlet transform sets up an isometry from $L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$ to $L^2(\mathbb{R}^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^n \times \mathrm{Spin}(n), Cl_{(p,q)})$. □

Theorem 2. *(Plancherel theorem). Let* $\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a, \mathbf{r}, s, b)$ *and* $\mathscr{C}\mathscr{S}_{\Psi}\mathbf{g}(a, \mathbf{r}, s, b)$ *be the Clifford-valued shearlet transforms of the multivector signals* $\mathbf{f}$ *and* $\mathbf{g}$, *respectively. Then, we have*

$$\begin{aligned} &\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)} Sc\left(\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a, \mathbf{r}, s, b)\overline{\mathscr{C}\mathscr{S}_{\Psi}\mathbf{g}(a, \mathbf{r}, s, b)}\right)\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}} \\ &= (2\pi)^n \int_{\mathbb{R}^{(p,q)}} Sc\left(\mathbf{f}(x)C_{\Psi}\overline{\mathbf{g}(x)}\right)\mathrm{d}^n x = (2\pi)^n \left|\langle \mathbf{f}C_{\Psi}, \mathbf{g}\rangle\right|_{L^2\left(\mathbb{R}^{(p,q)}, Cl_{(p,q)}\right)}, \end{aligned} \tag{36}$$

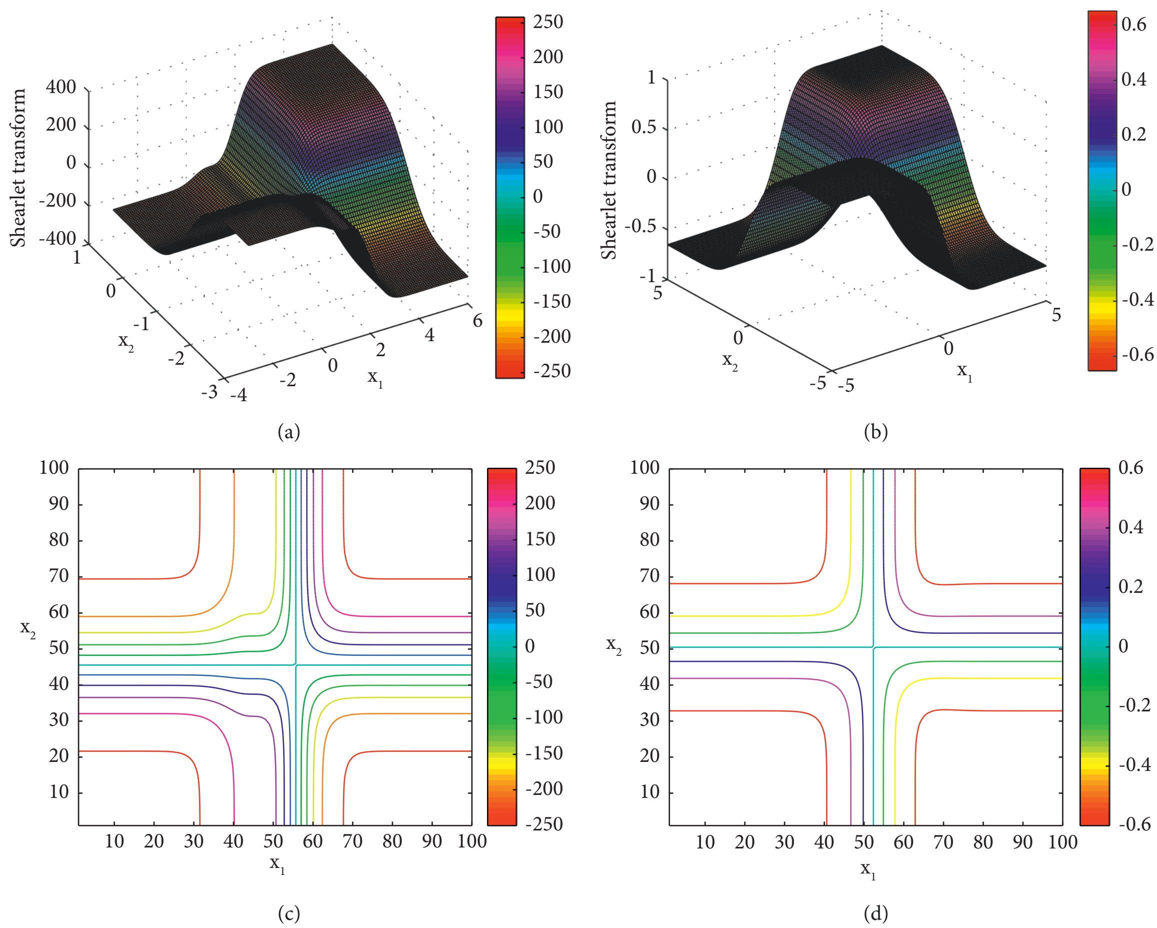

FIGURE 2: Two-dimensional Clifford-valued shearlet transforms of $\mathbf{f}(x_1,x_2)=\exp\{-(x_1^2+x_2^2)/2\}$ with respect to analyzing function $\Psi^{\mathbf{r}}_{a,s,b}(x_1,x_2)$ given by equation (35). (a) Clifford-valued ST of f at $a=1/2$, $b=1$, and $s=1/2$. (b) Clifford-valued ST of f at $a=b=1$, and $s=1$. (c) Contour plot of Clifford-valued ST of f at $a=1/2$, $b=1$, and $s=1/2$. (d) Contour plot of Clifford-valued ST of f at $a=b=1$, and $s=1$.

where C_Ψ is given by equation (25).

Proof. .Invoking the spectral representation (29) of Clifford shearlet transforms, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)} Sc\Big(\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\overline{\mathscr{C}\mathscr{S}_\Psi\mathbf{g}(a,\mathbf{r},s,b)}\Big)\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\\
&=\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)} a^{2-(1/n)}Sc\Big(\int_{\mathbb{R}^n}\mathscr{F}_{Cl}[\mathbf{f}](\xi)e^{I\upsilon(\xi,b)}\overline{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})}\mathrm{d}^n\xi\\
&\times\int_{\mathbb{R}^n}\overline{\mathscr{F}_{Cl}[\mathbf{g}](\xi')e^{I\upsilon(\xi',b)}\overline{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi'\overline{\mathbf{r}})}}\mathrm{d}^n\xi'\Big)\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}},\\
&=\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)} a^{2-(1/n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}Sc\Big(\mathscr{F}_{Cl}[\mathbf{f}](\xi)e^{I\upsilon(\xi,b)}\overline{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})}\\
&\times\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi'\overline{\mathbf{r}})e^{-I\upsilon(\xi',b)}\overline{\mathscr{F}_{Cl}[\mathbf{g}](\xi')}\Big)\mathrm{d}^n\xi\mathrm{d}^n\xi'\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}.
\end{aligned}\tag{37}$$

Then, equation (37) can be rewritten as

$$\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)} Sc\left(\mathscr{C}\mathscr{S}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\overline{\mathscr{C}\mathscr{S}_\Psi \mathbf{g}(a,\mathbf{r},s,b)}\right)\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}$$
$$= \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} Sc\Bigg(\mathscr{F}_{Cl}[\mathbf{f}](\xi)e^{Iv(\xi,b)}e^{-Iv(\xi',b)}$$
$$\times\overline{\{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})\}}\{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi'\overline{\mathbf{r}})\}\overline{\mathscr{F}_{Cl}[\mathbf{g}](\xi')}\Bigg)\mathrm{d}^n\xi\mathrm{d}^n\xi'\frac{\mathrm{d}a\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b}{a^{(n^2-n+1/n)}}\Bigg)$$
$$= (2\pi)^n\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} Sc\Bigg(\mathscr{F}_{Cl}[\mathbf{f}](\xi)\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{Iv(\xi-\xi',b)}\mathrm{d}^n b\Bigg)$$
$$\times\int_{\mathbb{R}^+\times\mathrm{Spin}(n)\times\mathbb{R}^{n-1}}\overline{\{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})\}}\{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi'\overline{\mathbf{r}})\}\frac{\mathrm{d}a\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s}{a^{(n^2-n+1/n)}}\times\overline{\mathscr{F}_{Cl}[\mathbf{g}](\xi')}\Big)\mathrm{d}^n\xi\mathrm{d}^n\xi'$$
$$= (2\pi)^n\Big[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} Sc\left(\mathscr{F}_{Cl}[\mathbf{f}](\xi)\delta(\xi-\xi')\right.$$
$$\times\int_{\mathbb{R}^+\times\mathrm{Spin}(n)\times\mathbb{R}^{n-1}}\overline{\{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})\}}\{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi'\overline{\mathbf{r}})\}\frac{\mathrm{d}a\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s}{a^{(n^2-n+1/n)}}\times\overline{\mathscr{F}_{Cl}[\mathbf{g}](\xi')}\Big)\mathrm{d}^n\xi\mathrm{d}^n\xi'$$
$$= (2\pi)^n\int_{\mathbb{R}^n} Sc\Bigg(\mathscr{F}_{Cl}[\mathbf{f}](\xi)\times\int_{\mathbb{R}^+\times\mathrm{Spin}(n)\times\mathbb{R}^{n-1}}\overline{\{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})\}}\{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})\}\frac{\mathrm{d}a\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s}{a^{(n^2-n+1/n)}}\times\overline{\mathscr{F}_{Cl}[\mathbf{g}](\xi)}\Bigg)\mathrm{d}^n\xi$$
$$= (2\pi)^n\int_{\mathbb{R}^n} Sc\left(\mathscr{F}_{Cl}[\mathbf{f}](\xi)C_\Psi\overline{\mathscr{F}_{Cl}[\mathbf{g}](\xi)}\right)\mathrm{d}^n\xi$$
$$= (2\pi)^n\int_{\mathbb{R}^{(p,q)}} Sc\left(\mathbf{f}(x)C_\Psi\overline{\mathbf{g}(x)}\right)\mathrm{d}^n x. \tag{38}$$

This completes the proof of Theorem 2. □

Corollary 1. *For* $\mathbf{f}=\mathbf{g}$, *we have the following identity:*

$$\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)}\left|\mathscr{C}\mathscr{S}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\right|^2\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}} = (2\pi)^n\int_{\mathbb{R}^{(p,q)}} Sc\left(\mathbf{f}(x)C_\Psi\overline{\mathbf{f}(x)}\right)\mathrm{d}^n x. \tag{39}$$

By taking $\Psi\in L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})$ with $C_\Psi=1$, the Clifford-valued shearlet transform $\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)$ becomes an isometry from $L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})$ to $L^2(\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n),Cl_{(p,q)})$.

The next theorem guarantees the reconstruction of the input Clifford-valued signal from the corresponding Clifford-valued shearlet transform.

Theorem 3 (Inversion formula). *Any Clifford-valued signal* $\mathbf{f}\in L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})$ *can be reconstructed from the Clifford-valued shearlet transform* $\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)$ *via the formula:*

$$\mathbf{f}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)}\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\Psi^{\mathbf{r}}_{a,s,b}(x)C_\Psi^{-1}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}. \tag{40}$$

Proof. Implication of Plancherel theorem of Clifford-valued shearlet transform (36) for every $g \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$ yields that

$$\begin{aligned}(2\pi)^n\left|\langle \mathbf{f}C_\Psi, \mathbf{g}\rangle\right|_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)} &= \left|\langle \mathscr{C}\mathscr{S}_\Psi\mathbf{f}, \mathscr{C}\mathscr{S}_\Psi\mathbf{g}\rangle\right|_{L^2\left(\mathscr{G},Cl_{(p,q)}\right)}\\ &= \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)} Sc\Big(\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\overline{\mathscr{C}\mathscr{S}_\Psi\mathbf{g}(a,\mathbf{r},s,b)}\Big)\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\\ &= \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)} Sc\Big(\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\overline{\int_{\mathbb{R}^{(p,q)}}\mathbf{g}(x)\overline{\Psi^{\mathbf{r}}_{a,s,b}(x)}}\Big)\mathrm{d}^n x\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\\ &= \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)}\int_{\mathbb{R}^{(p,q)}} Sc\Big(\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\Psi^{\mathbf{r}}_{a,s,b}(x)\overline{\mathbf{g}(x)}\Big)\mathrm{d}^n x\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\\ &= \int_{\mathbb{R}^{(p,q)}}\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)} Sc\Big(\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\Psi^{\mathbf{r}}_{a,s,b}(x)\Big)\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\overline{\mathbf{g}(x)}\mathrm{d}^n x\\ &= \left|\left\langle \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)} \mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\Psi^{\mathbf{r}}_{a,s,b}(x)\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}, \mathbf{g}\right\rangle\right|_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)},\end{aligned} \tag{41}$$

where we used the Fubini–Tonelli theorem in getting the second last step. Since $\mathbf{g} \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$ is arbitrary, we have

$$(2\pi)^n\mathbf{f}(x)C_\Psi = \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)} \mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\Psi^{\mathbf{r}}_{a,s,b}(x)\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}, \tag{42}$$

or equivalently

$$\mathbf{f}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathrm{Spin}(n)} \mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\Psi^{\mathbf{r}}_{a,s,b}(x)C_\Psi^{-1}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}. \tag{43}$$

This completes the proof of Theorem 3.

The next theorem presents a characterization of the range of the Clifford-valued shearlet transform $\mathscr{C}\mathscr{S}\mathscr{H}_\Psi$. The result follows as a consequence of the reconstruction formula (40) and the well known Fubini theorem. □

Theorem 4 (Characterization of range of $\mathscr{C}\mathscr{S}_\Psi$). *If $\mathbf{h} = \mathscr{C}\mathscr{S}_\Psi\mathbf{f} \in L^2(\mathscr{G}, Cl_{(p,q)})$, let Ψ be an admissible Clifford-valued shearlet. Then, $\mathbf{h}$ is a Clifford-valued shearlet transform of a function $\mathbf{f} \in L^2(\mathscr{G}, Cl_{(p,q)})$ if and only if it satisfies the reproducing property:*

$$\mathbf{h}(a',\mathbf{r}',s',b') = \frac{1}{(2\pi)^n}\int_{\mathscr{G}}\mathbf{h}(a,\mathbf{r},s,b)\langle\Psi^{\mathbf{r}}_{a,s,b}C_\Psi^{-1}, \Psi^{\mathbf{r}'}_{a',s',b'}\rangle_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}. \tag{44}$$

Proof. Let $\mathbf{h}$ belongs to the range of the Clifford-valued shearlet transform $\mathscr{C}\mathscr{S}_\Psi$. Then, there exist a Clifford-valued function $\mathbf{f} \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$ such that $\mathscr{C}\mathscr{S}_\Psi\mathbf{f} = \mathbf{h}$. In order to show that $\mathbf{h}$ satisfies equation (44), we proceed as

$$\begin{aligned}\mathbf{h}(a',\mathbf{r}',s',b') &= \mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a',\mathbf{r}',s',b')\\ &= \int_{\mathbb{R}^{(p,q)}} \mathbf{f}(x)\overline{\Psi^{\mathbf{r}'}_{a',s',b'}}\,\mathrm{d}^n x\\ &= \int_{\mathbb{R}^{(p,q)}} \left[\frac{1}{(2\pi)^n}\int_{\mathscr{G}} \mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\Psi^{\mathbf{r}}_{a,s,b}(x)C_{\Psi}^{-1}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\right]\overline{\Psi^{\mathbf{r}'}_{a',s',b'}}\,\mathrm{d}^n x\\ &= \frac{1}{(2\pi)^n}\int_{\mathscr{G}} \mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\left[\int_{\mathbb{R}^{(p,q)}} \Psi^{\mathbf{r}}_{a,s,b}(x)C_{\Psi}^{-1}\overline{\Psi^{\mathbf{r}'}_{a',s',b'}}\,\mathrm{d}^n x\right]\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\\ &= \frac{1}{(2\pi)^n}\int_{\mathscr{G}} \mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\langle \Psi^{\mathbf{r}}_{a,s,b}C_{\Psi}^{-1},\Psi^{\mathbf{r}'}_{a',s',b'}\rangle_{L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\\ &= \frac{1}{(2\pi)^n}\int_{\mathscr{G}} \mathbf{h}(a,\mathbf{r},s,b)\langle \Psi^{\mathbf{r}}_{a,s,b}C_{\Psi}^{-1},\Psi^{\mathbf{r}'}_{a',s',b'}\rangle_{L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}.\end{aligned} \tag{45}$$

Conversely, suppose that an arbitrary function $\mathbf{h}\in L^2(\mathscr{G},Cl_{(p,q)})$ satisfies equation (44). Then, we show that there exists $\mathbf{f}\in L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})$, such that $\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}=\mathbf{h}$. Assume that

$$\mathbf{f}(x) = \frac{1}{(2\pi)^n}\int_{\mathscr{G}} \mathbf{h}(a,\mathbf{r},s,b)\Psi^{\mathbf{r}}_{a,s,b}(x)C_{\Psi}^{-1}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}. \tag{46}$$

Then, it can be easily verified that

$$\|\mathbf{f}\|^2_{L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})} = \frac{1}{(2\pi)^{2n}}\left|C_{\Psi}^{-1}\right|^2\left\|\Psi^{\mathbf{r}}_{a,s,b}\right\|^2_{L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})}\|\mathbf{h}(a,\mathbf{r},s,b)\|^2_{L^2(\mathscr{G},Cl_{(p,q)})}, \tag{47}$$

which implies that $\mathbf{f}\in L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})$. Moreover, as a consequence of the well-known Fubini theorem and inversion Theorem (40), we have

$$\begin{aligned}\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a',\mathbf{r}',s',b') &= \int_{\mathbb{R}^{(p,q)}} \mathbf{f}(x)\overline{\Psi^{\mathbf{r}'}_{a',s',b'}(x)}\,\mathrm{d}^n x\\ &= \int_{\mathbb{R}^{(p,q)}} \left[\frac{1}{(2\pi)^n}\int_{\mathscr{G}} \mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\Psi^{\mathbf{r}}_{a,s,b}C_{\Psi}^{-1}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\right]\overline{\Psi^{\mathbf{r}'}_{a',s',b'}(x)}\,\mathrm{d}^n x\\ &= \frac{1}{(2\pi)^n}\int_{\mathscr{G}} \mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\int_{\mathbb{R}^{(p,q)}} \Psi^{\mathbf{r}}_{a,s,b}C_{\Psi}^{-1}\overline{\Psi^{\mathbf{r}'}_{a',s',b'}(x)}\,\mathrm{d}^n x\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\\ &= \frac{1}{(2\pi)^n}\int_{\mathscr{G}} \mathbf{h}(a,\mathbf{r},s,b)\langle \Psi^{\mathbf{r}}_{a,s,b}C_{\Psi}^{-1},\Psi^{\mathbf{r}'}_{a',s',b'}\rangle_{L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\\ &= \mathbf{h}(a',\mathbf{r}',s',b').\end{aligned} \tag{48}$$

This evidently completes the proof of theorem. □

Corollary 2. *For an admissible Clifford shearlet $\Psi\in L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})$, the range of the Clifford shearlet transform 28) is a reproducing kernel in $L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})$ with kernel that can be given by*

$$K_{\Psi}(a,\mathbf{r},s,b,a',\mathbf{r}',s',b') = \frac{1}{(2\pi)^n}\langle \Psi^{\mathbf{r}}_{a,s,b}C_{\Psi}^{-1},\Psi^{\mathbf{r}'}_{a',s',b'}\rangle_{L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})}. \tag{49}$$

4. HAP Property for the Clifford-Valued Shearlet Transforms

Homogeneous approximation property (HAP) means that the approximation rate in a reconstruction of signal is essentially invariant under time-scale shifts. The HAP is being extensively used for studying frame density [17]. In this section, we investigate the homogeneous approximation property for the proposed Clifford-valued shearlet transforms. Initially, we shall present some results related to the pointwise convergence of the reconstruction formula (40).

Theorem 5. *Let $\mathcal{CS}_\Psi \mathbf{f}(a,\mathbf{r},s,b)$ be the Clifford-valued shearlet transform of any $\mathbf{f} \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$ such that*

$$\mathbf{f}_{M,N}(x) = \frac{1}{(2\pi)^n}\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \mathcal{CS}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\Psi^{\mathbf{r}}_{a,s,b}(x) C_\Psi^{-1}\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{n+1}}, \quad N > M > 0, \tag{50}$$

where $\Psi \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$ is an admissible Clifford-valued shearlet with $C_\Psi \neq 0$, real valued. Then, we have

$$\mathcal{F}_{Cl}\left[\mathbf{f}_{M,N}\right](\xi) = \mathcal{F}_{Cl}[\mathbf{f}](\xi)\int_M^N \int_{\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \left|\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}]\left(\mathbf{r}S_s A_a \xi' \overline{\mathbf{r}}\right)\right|^2 C_\Psi^{-1}\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{\left(n^2-n+1/n\right)}}. \tag{51}$$

Proof. For $M, N \in \mathbb{R}^+$, we define

$$\mathbf{f}_{M,N}(x) = \frac{1}{(2\pi)^n}\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \mathcal{CS}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\Psi^{\mathbf{r}}_{a,s,b}(x) C_\Psi^{-1}\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{n+1}}. \tag{52}$$

Then, the application of Schwartz's inequality implies that

$$\begin{aligned}
&\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \left|\mathcal{CS}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\Psi^{\mathbf{r}}_{a,s,b}(x)\right| C_\Psi^{-1}\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{n+1}} \leq \int_M^N \left\{\int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \left|\mathcal{CS}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\right|^2 \mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\right\}^{1/2}\\
&\quad\times\left\{\int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \left|\Psi^{\mathbf{r}}_{a,s,b}(x)\right|^2 \mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\right\}^{1/2} C_\Psi^{-1}\frac{\mathrm{d}a}{a^{n+1}}\\
&= \int_M^N \left\{\int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \left|\mathcal{CS}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\right|^2 \mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\right\}^{1/2}\\
&\quad\times\left\|\Psi^{\mathbf{r}}_{a,s,b}(x)\right\|_{L^2\left(R^n\times R^{n-1}\times \mathrm{Spin}(n)\right)} C_\Psi^{-1}\frac{\mathrm{d}a}{a^{n+1}}\\
&\leq \left\{\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \left|\mathcal{CS}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\right|^2 \mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\frac{\mathrm{d}a}{a^{n+1}}\right\}^{1/2}\\
&\times\|\Psi\|_{L^2\left(R^n\times R^{n-1}\times \mathrm{Spin}(n)\right)} C_\Psi^{-1}\left\{\int_M^N \frac{\mathrm{d}a}{a^{n+1}}\right\}^{1/2} \leq \left\{(2\pi)^n\left|\langle \mathbf{f}C_\Psi, \mathbf{f}\rangle\right|_{L^2\left(\mathbb{R}^{(p,q)}, Cl_{(p,q)}\right)}\right\}^{1/2}\\
&\quad\|\Psi\|_{L^2\left(R^n\times R^{n-1}\times \mathrm{Spin}(n)\right)} C_\Psi^{-1}\frac{1}{\sqrt{n}}\left[M^{-n} - N^{-n}\right]^{1/2} < \infty.
\end{aligned} \tag{53}$$

This shows that $\mathbf{f}_{M,N}$ is well defined on $\mathbb{R}^2$.

Next, we show that $\mathbf{f}_{M,N}$ is uniformly continuous on $\mathbb{R}^n$. For any $x, x' \in \mathbb{R}^n$, we have

$$\begin{aligned}
\left|\mathbf{f}_{M,N}(x) - \mathbf{f}_{M,N}(x')\right| &= \left|\frac{1}{(2\pi)^n}\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \mathscr{C}\mathscr{S}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\left[\Psi^{\mathbf{r}}_{a,s,b}(x) - \Psi^{\mathbf{r}}_{a,s,b}(x')\right] C_\Psi^{-1}\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{n+1}}\right| \\
&\le \left\{\frac{1}{(2\pi)^n}\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \left|\mathscr{C}\mathscr{S}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\right|^2 \frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{n+1}}\right\}^{1/2} \\
&\quad\times \left\{\frac{1}{(2\pi)^n}\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \left|\Psi^{\mathbf{r}}_{a,s,b}(x) - \Psi^{\mathbf{r}}_{a,s,b}(x')\right|^2 \frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{n+1}}\right\}^{1/2} \left|C_\Psi^{-1}\right| \\
&\le \frac{1}{(2\pi)^{n/2}}\left\{\left|\langle \mathbf{f}C_\Psi, \mathbf{f}\rangle\right|_{L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})}\right\}^{1/2} \times \left\{\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \left|\Psi^{\mathbf{r}}_{a,s,b}(x) - \Psi^{\mathbf{r}}_{a,s,b}(x')\right|^2 \frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{n+1}}\right\}^{1/2} \left|C_\Psi^{-1}\right|.
\end{aligned} \tag{54}$$

From equation (54), we observe that $\left|\mathbf{f}_{M,N}(\mathbf{x}) - \mathbf{f}_{M,N}(\mathbf{x}')\right| \longrightarrow 0$ as $\|x - x'\| \longrightarrow 0$. Thus, we conclude that $\mathbf{f}_{M,N}$ is uniformly continuous on $\mathbb{R}^{(p,q)}$.

Moreover, for any $\mathbf{g} \in L^1 \cap L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$, we have

$$\begin{aligned}
\left|\langle \mathbf{f}_{M,N}, \mathbf{g}\rangle\right|_{L^2(\mathbb{R}^n, Cl_{(p,q)})} &= \int_{\mathbb{R}^n} Sc\left(\mathbf{f}_{M,N}(x)\overline{\mathbf{g}(x)}\right)\mathrm{d}^n x \\
&= Sc\left(\int_{\mathbb{R}^n}\left\{\frac{1}{(2\pi)^n}\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \mathscr{C}\mathscr{S}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\Psi^{\mathbf{r}}_{a,s,b}(x) C_\Psi^{-1}\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{n+1}}\right\}\overline{\mathbf{g}(x)}\right)\mathrm{d}^n x \\
&= \frac{1}{(2\pi)^n}\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} Sc\left(\mathscr{C}\mathscr{S}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\left\{\int_{\mathbb{R}^n}\Psi^{\mathbf{r}}_{a,s,b}(x)\overline{\mathbf{g}(x)}\mathrm{d}^n x\right\}C_\Psi^{-1}\right)\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{n+1}} \\
&= \frac{1}{(2\pi)^n}\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} Sc\left(\mathscr{C}\mathscr{S}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\overline{\mathscr{C}\mathscr{S}_\Psi \mathbf{g}(a,\mathbf{r},s,b)}\mathrm{d}^n x C_\Psi^{-1}\right)\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{n+1}} \\
&= \frac{1}{(2\pi)^n} Sc\left(\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)}\left\{a^{1-(1/2n)}\int_{\mathbb{R}^n}\mathscr{F}_{Cl}[\mathbf{f}](\xi)e^{I\upsilon(\xi,b)}\overline{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_s A_a \xi\overline{\mathbf{r}})}\mathrm{d}^n\xi\right\}\right. \\
&\quad\left.\times\overline{\left\{a^{1-(1/2n)}\int_{\mathbb{R}^n}\mathscr{F}_{Cl}[\mathbf{g}](\xi')e^{I\upsilon(\xi,b)}\overline{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_s A_a \xi'\overline{\mathbf{r}})}\mathrm{d}^n\xi'\right\}}C_\Psi^{-1}\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{n+1}}\right) \\
&= \frac{1}{(2\pi)^n} Sc\left(\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)}\int_{\mathbb{R}^n}\mathscr{F}_{Cl}[\mathbf{f}](\xi)e^{I\upsilon(\xi,b)}\overline{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_s A_a \xi\overline{\mathbf{r}})}\mathrm{d}^n\xi\right. \\
&\quad\left.\times\int_{\mathbb{R}^n}\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_s A_a \xi'\overline{\mathbf{r}})e^{-I\upsilon(\xi',b)}\overline{\mathscr{F}_{Cl}[\mathbf{g}](\xi')}\mathrm{d}^n\xi' C_\Psi^{-1}\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{(n^2-n+1/n)}}\right) \\
&= \frac{1}{(2\pi)^n} Sc\left(\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times \mathrm{Spin}(n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathscr{F}_{Cl}[\mathbf{f}](\xi)e^{I\upsilon(\xi,b)}e^{-I\upsilon(\xi',b)}\overline{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_s A_a \xi\overline{\mathbf{r}})}\mathrm{d}^n\xi\right. \\
&\quad\left.\times\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_s A_a \xi'\overline{\mathbf{r}})\overline{\mathscr{F}_{Cl}[\mathbf{g}](\xi')}\mathrm{d}^n\xi' C_\Psi^{-1}\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{(n^2-n+1/n)}}\right)
\end{aligned}$$

$$= Sc\left(\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times\text{Spin}(n)} \int_{\mathbb{R}^n} \mathcal{F}_{Cl}[\mathbf{f}](\xi)\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{Iv(\xi-\xi',b)}\mathrm{d}^n b\overline{\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})}\mathrm{d}^n\xi \times \mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})\overline{\mathcal{F}_{Cl}[\mathbf{g}](\xi')}\mathrm{d}^n\xi' C_\Psi^{-1}\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}a}{a^{(n^2-n+1/n)}}\right)$$
$$= Sc\left(\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times\text{Spin}(n)} \int_{\mathbb{R}^n} \mathcal{F}_{Cl}[\mathbf{f}](\xi)\delta(\xi-\xi')\overline{\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})} \times \mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi'\overline{\mathbf{r}})\overline{\mathcal{F}_{Cl}[\mathbf{g}](\xi')}\mathrm{d}^n\xi' C_\Psi^{-1}\right)\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}}$$
$$= Sc\left(\int_M^N \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}\times\text{Spin}(n)} \int_{\mathbb{R}^n} \mathcal{F}_{Cl}[\mathbf{f}](\xi)\overline{\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})} \overline{\times\mathcal{F}_{Cl}[\mathbf{g}](\xi)\overline{\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})}}\mathrm{d}^n\xi C_\Psi^{-1}\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}}\right)$$
$$= \int_M^N \int_{\mathbb{R}^{n-1}\times\text{Spin}(n)} \int_{\mathbb{R}^n} Sc\left(\mathcal{F}_{Cl}[\mathbf{f}](\xi)\left|\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})\right|^2\overline{\mathcal{F}_{Cl}[\mathbf{g}](\xi)}\mathrm{d}^n\xi C_\Psi^{-1}\right)\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}}$$
$$= \int_{\mathbb{R}^n} Sc\left(\left\{\mathcal{F}_{Cl}[\mathbf{f}](\xi)\int_M^N \int_{\mathbb{R}^{n-1}\times\text{Spin}(n)} \left|\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})\right|^2 C_\Psi^{-1}\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}}\right\}\overline{\mathcal{F}_{Cl}[\mathbf{g}](\xi)}\right)\mathrm{d}^n\xi. \tag{55}$$

Invoking scalar part for the Clifford Fourier transform, we can deduce that

$$\mathcal{F}\left[\mathbf{f}_{M,N}\right](\xi) = \mathcal{F}_{Cl}[\mathbf{f}](\xi)\int_M^N \int_{\mathbb{R}^{n-1}\times\text{Spin}(n)} \left|\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})\right|^2 C_\Psi^{-1}\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}}. \tag{56}$$

This completes the proof of Theorem 5. □

Theorem 6. *Let $\Psi \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$ be an admissible Clifford-valued shearlet. Then, for any $\mathbf{f} \in L^1 \cap L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$, we have*

$$\lim_{M\longrightarrow 0}, N \longrightarrow \infty \left\|\mathbf{f} - \mathbf{f}_{M,N}\right\|_\infty = 0, \qquad \lim_{M\longrightarrow 0}, N \longrightarrow \infty \left\|\mathbf{f} - \mathbf{f}_{M,N}\right\|_2 = 0. \tag{57}$$

Proof. Using Parseval's formula for the Clifford Fourier transforms together with an application of Theorem 5, we have

$$\begin{aligned}\left\|\mathbf{f} - \mathbf{f}_{M,N}\right\|_{L^\infty(\mathbb{R}^n, Cl_{(p,q)})} &\le \left\|\mathbf{f} - \mathbf{f}_{M,N}\right\|_{L^1(\mathbb{R}^n, Cl_{(p,q)})}\\ &= \left\|\mathcal{F}_{Cl}[\mathbf{f}](\xi) - \mathcal{F}_{Cl}\left[\mathbf{f}_{M,N}\right](\xi)\right\|_{L^1(\mathbb{R}^n, Cl_{(p,q)})}\\ &= \|\mathcal{F}_{Cl}[\mathbf{f}](\xi) - \left\{\mathcal{F}_{Cl}[\mathbf{f}](\xi)\int_M^N\int_{\mathbb{R}^{n-1}} \times \text{Spin}(n)\times\left|\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})\right|^2\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}}C_\Psi^{-1}\right\}\|_{L^1(\mathbb{R}^n, Cl_{(p,q)})}\\ &= \|\mathcal{F}_{Cl}[\mathbf{f}](\xi)\left\{1 - \int_M^N\int_{\mathbb{R}^{n-1}} \times \text{Spin}(n)\times\left|\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})\right|^2\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}}C_\Psi^{-1}\right\}\|_{L^1(\mathbb{R}^n, Cl_{(p,q)})}\\ &= \int_{\mathbb{R}^n}\left|\mathcal{F}_{Cl}[\mathbf{f}](\xi)\right|\|1 - \int_M^N\int_{\mathbb{R}^{n-1}}\times\text{Spin}(n)\times\left|\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})\right|^2\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}}C_\Psi^{-1}|\mathrm{d}^n\xi.\end{aligned} \tag{58}$$

Since Ψ is given to be admissible, it follows that

$$\begin{aligned}&\int_M^N \int_{\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \left|\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_s A_a \xi \overline{\mathbf{r}})\right|^2 \frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}}\\ &\leq \int_{\mathbb{R}^+} \int_{\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \left|\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_s A_a \xi \overline{\mathbf{r}})\right|^2 \frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}} = C_\Psi < \infty.\end{aligned} \tag{59}$$

Therefore, we have

$$\lim_{M\longrightarrow 0}, N \longrightarrow \infty \left|1 - \int_M^N \int_{\mathbb{R}^{n-1}\times \mathrm{Spin}(n)} \mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_s A_a \xi \overline{\mathbf{r}})|^2 \frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}} C_\Psi^{-1}\right| = 0. \tag{60}$$

Using dominated convergence theorem in equation (58), we conclude that

$$\lim_{M\longrightarrow 0}, N \longrightarrow \infty \left\|\mathbf{f} - \mathbf{f}_{M,N}\right\|_{L^\infty(\mathbb{R}^n, Cl_{(p,q)})} = 0. \tag{61}$$

Proceeding in a manner similar to the above case, we can show that

$$\lim_{M\longrightarrow 0}, N \longrightarrow \infty \left\|\mathbf{f} - \mathbf{f}_{M,N}\right\|_{L^2(\mathbb{R}^n, Cl_{(p,q)})} = 0. \tag{62}$$

This completes the proof of Theorem 6.

In the sequel, we study the homogeneous approximation property for the proposed Clifford-valued shearlet transforms. Prior to that, we introduce some notations as given below:

For every $(a', \mathbf{r}', s', b') \in L^2(\mathbb{R}^+ \times \mathrm{Spin}(n) \times \mathbb{R}^{n-1} \times \mathbb{R}^n, Cl_{(p,q)})$ and $M > N, P > 0$, we denote

$$\begin{aligned}Q_{M,N;P} &= ([-N,-M]\cup[N,M]) \times \mathrm{Spin}(n) \times [-P,P]^{n-1} \times [-P,P]^n,\\ (a', s', b', \mathbf{r}')Q_{M,N;P} &= \{(a', s', b', \mathbf{r}')(a, s, b, \mathbf{r})\\ &= \left(a'a, s' + a'^{1-(1/n)}s + s', b' + S_{s'}A_{a'}b, \mathbf{r}'\mathbf{r}\right)\},\end{aligned} \tag{63}$$

where $a \in [-N,-M]\cup[N,M], \mathbf{r} \in \mathrm{Spin}(n), s \in [-P,P]^{n-1}$ and $b \in [-P,P]^n$. □

Theorem 7. *Let $\Psi \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$be an admissible Clifford-valued shearlet with $C_\Psi \neq 0$, real valued. Then, for any $\mathbf{f} \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$ and $\varepsilon > 0$, there exist some constants $N > M > 0, P > 0$, such that for any $(a', \mathbf{r}', s', b') \in L^2(\mathbb{R}^+ \times Spin(n) \times \mathbb{R}^{n-1} \times \mathbb{R}^n, Cl_{(p,q)})$, with any $0 < M' \leq M, N \leq N'$ and $P' \geq P$, we have*

$$\left\|\mathbf{f}^{\mathbf{r}'}_{a',s',b'} - \int_{(a,s,b,\mathbf{r})\in\mathscr{Q}} \langle \mathbf{f}^{\mathbf{r}'}_{a',s',b'}, \Psi^{\mathbf{r}}_{a,s,b}\rangle C_\Psi^{-1} \Psi^{\mathbf{r}}_{a,s,b} \frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\right\|^2_{L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})} < \varepsilon, \tag{64}$$

where $(a', s', b', \mathbf{r}')Q_{M',N';P'} = \mathscr{Q}'$.

Proof. For an arbitrary $\mathbf{g} \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$, we have

$$\begin{aligned}
&\left\| \mathbf{f}_{a'}, s', b'^{\mathbf{r}} - \int_{(a,s,b,\mathbf{r})\in \mathcal{Q}'} \langle \mathbf{f}_{a'}, s', b'^{\mathbf{r}}, \Psi^{\mathbf{r}}_{a,s,b}\rangle C_\Psi^{-1}\Psi^{\mathbf{r}}_{a,s,b}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\right\|^2_{L^2\mathbb{R}^{(p,q)},Cl_{(p,q)}}\\
&= \sup_{\|\mathbf{g}\|=1}\left|\langle \mathbf{f}_{a'}, s', b'^{\mathbf{r}} - \int_{(a,s,b,\mathbf{r})\in \mathcal{Q}'} \langle \mathbf{f}_{a'}, s', b'^{\mathbf{r}}, \Psi^{\mathbf{r}}_{a,s,b}\rangle C_\Psi^{-1}\Psi^{\mathbf{r}}_{a,s,b}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}, \mathbf{g}\rangle\right|^2\\
&= \sup_{\|\mathbf{g}\|=1}\left|\langle \mathbf{f}_{a'}, s', b'^{\mathbf{r}}, \mathbf{g}\rangle \left\langle \int_{(a,s,b,\mathbf{r})\in \mathcal{Q}'} \langle \mathbf{f}_{a'}, s', b'^{\mathbf{r}}, \Psi^{\mathbf{r}}_{a,s,b}\rangle C_\Psi^{-1}\Psi^{\mathbf{r}}_{a,s,b}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}, \mathbf{g}\right\rangle\right|^2\\
&= \sup_{\|\mathbf{g}\|=1}\left|\langle \mathbf{f}_{a'}, s', b'^{\mathbf{r}}, \mathbf{g}\rangle - \int_{(a,s,b,\mathbf{r})\in \mathcal{Q}'} \langle \mathbf{f}_{a'}, s', b'^{\mathbf{r}}, \Psi^{\mathbf{r}}_{a,s,b}\rangle C_\Psi^{-1}\Psi^{\mathbf{r}}_{a,s,b}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\right|^2\\
&= \sup_{\|\mathbf{g}\|=1}\left|\int_{(a,s,b,\mathbf{r})\notin \mathcal{Q}'} \langle \mathbf{f}_{a'}, s', b'^{\mathbf{r}}, \Psi^{\mathbf{r}}_{a,s,b}\rangle C_\Psi^{-1}\Psi^{\mathbf{r}}_{a,s,b}\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\right|^2\\
&\leq \sup_{\|\mathbf{g}\|=1}\int_{(a,s,b,\mathbf{r})\notin \mathcal{Q}'} \langle \mathbf{f}_{a'}, s', b'^{\mathbf{r}}, \Psi^{\mathbf{r}}_{a,s,b}\rangle \left|C_\Psi^{-1}\right|^2\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}} \times \int_{(a,s,b,\mathbf{r})\notin \mathcal{Q}'}\left|\langle \Psi^{\mathbf{r}}_{a,s,b}, \mathbf{g}\rangle^2\right|\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\\
&= \int_{(a,s,b,\mathbf{r})\notin \mathcal{Q}'}\left|\langle \mathbf{f}_{a'}, s', b'^{\mathbf{r}}, \Psi^{\mathbf{r}}_{a,s,b}\rangle\right|^2 \left|C_\Psi^{-1}\right|^2\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}} \times \sup_{\|\mathbf{g}\|=1}\int_{(a,s,b,\mathbf{r})\notin \mathcal{Q}'}\left|\langle \mathbf{g}, \Psi^{\mathbf{r}}_{a,s,b}\rangle^2\right|\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\\
&= \int_{(a,s,b,\mathbf{r})\notin \mathcal{Q}'}\left|\langle \mathbf{f}_{a'}, s', b'^{\mathbf{r}}, \Psi^{\mathbf{r}}_{a,s,b}\rangle\right|^2 \left|C_\Psi^{-1}\right|^2\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}} \times \sup_{\|\mathbf{g}\|=1}\int_{(a,s,b,\mathbf{r})\notin \mathcal{Q}'}\left|\mathcal{CS}_\Psi \mathbf{g}(a,s,b,\mathbf{r})\right|^2\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\\
&= \int_{(a,s,b,\mathbf{r})\notin \mathcal{Q}'}\left|\langle \mathbf{f}_{a'}, s', b'^{\mathbf{r}}, \Psi^{\mathbf{r}}_{a,s,b}\rangle\right|^2 \left|C_\Psi^{-1}\right|^2\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}} \times C_\Psi\\
&= \int_{(a,s,b,\mathbf{r})\notin \mathcal{Q}_{M',N';P'}}\left|\langle \mathbf{f}, \Psi^{\mathbf{r}}_{a,s,b}\rangle\right|^2\frac{\mathrm{d}a\mathrm{d}^{n-1}s\mathrm{d}^n b\mathrm{d}\mathbf{r}}{a^{n+1}}\left|C_\Psi^{-1}\right|^2 \times C_\Psi.
\end{aligned} \tag{65}$$

By choosing N and P large enough and M arbitrary small, we can make R. H. S as small as we need. This completes the proof of Theorem 7. □

5. Uncertainty Principles for the Clifford-Valued Shearlet Transforms

In this section, we shall establish several uncertainty inequalities including Heisenberg–Pauli–Weyl uncertainty inequality, Pitt's inequality, and logarithmic and local uncertainty inequality for the Clifford-valued shearlet transform as defined by equation (28). Prior to establishing the uncertainty principle for the Clifford-valued shearlet transform, we have the following lemma which shall be employed for deriving certain uncertainty inequalities and whose proof follows directly from the Parseval's and inversion formulae of the Clifford Fourier transforms.

Lemma 1. *Let $\Psi \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$ be an admissible Clifford-valued shearlet. Then, for any $\mathbf{f} \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$, we have*

$$\mathcal{F}_{Cl}\left[\mathcal{CS}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\right](\xi) = (2\pi)^{(n/2)} a^{1-(1/2n)}\mathcal{F}_{Cl}[\mathbf{f}](\xi)\overline{\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}]\left(\mathbf{r}S_s A_a \xi\overline{\mathbf{r}}\right)}. \tag{66}$$

Theorem 8 (Heisenberg–Weyl inequality). *Let $\mathcal{CS}_\Psi \mathbf{f}(a,\mathbf{r},s,b)$ be the Clifford-valued shearlet transform of any Clifford-valued function $\mathbf{f} \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$. Then, the following inequality follows*

$$\left\|b\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right\|_{L^2\left(\mathscr{G},Cl_{(p,q)}\right)}\left\|\xi\mathscr{F}_{Cl}[\mathbf{f}](\xi)C_{\Psi}\right\|_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}\geq\frac{1}{2}\left|\langle\mathbf{f}(x)C_{\Psi},\mathbf{f}(x)\rangle\right|_{L^2\left(\mathscr{G},Cl_{(p,q)}\right)}. \tag{67}$$

Proof. For any Clifford-valued function $\mathbf{f}\in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$, the Heisenberg–Paul–Weyl inequality for the Clifford Fourier transforms [8, 18] is given by

$$\left\{\int_{\mathbb{R}^n}|b|^2|\mathbf{f}(b)|^2\mathrm{d}^n b\right\}^{1/2}\left\{\int_{\mathbb{R}^n}|\xi|^2\left|\mathscr{F}_{Cl}[\mathbf{f}](\xi)\right|\mathrm{d}^n\xi\right\}^{1/2}\geq\frac{1}{2(2\pi)^{n/2}}\int_{\mathbb{R}^n}|\mathbf{f}(b)|^2\mathrm{d}^n b. \tag{68}$$

Considering $\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)$ as a function of b and replacing $\mathbf{f}$ by $\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)$ in (68), we get

$$\left\{\int_{\mathbb{R}^n}|b|^2\left|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}^n b\right\}^{1/2}\left\{\int_{\mathbb{R}^n}|\xi|^2\left|\mathscr{F}_{Cl}\left[\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right](\xi)\right|^2\mathrm{d}^n\xi\right\}^{1/2}\geq\frac{1}{2(2\pi)^{(n/2)}}\int_{\mathbb{R}^n}\left|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}^n b. \tag{69}$$

We now integrate the above inequality with respect to measure $(\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a/a^{n+1})$, and using Schwartz inequality, to obtain

$$\begin{aligned}&\left\{\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\int_{\mathbb{R}^n}|b|^2\left|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}^n b\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}\right\}^{1/2}\times\left\{\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\int_{\mathbb{R}^n}|\xi|^2\left|\mathscr{F}_c\left[\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right](\xi)\right|^2\mathrm{d}^n\xi\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}\right\}^{1/2}\\&\geq\frac{1}{2(2\pi)^{(n/2)}}\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\int_{\mathbb{R}^n}\left|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right|^2\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}\mathrm{d}^n b.\end{aligned} \tag{70}$$

Using Lemma 1 together with Fubini theorem, we obtain

$$\begin{aligned}&\left\{\int_{\mathbb{R}^n\times\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}|b|^2\left|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}^n b\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}\right\}^{(1/2)}\\&\times\left\{\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\int_{\mathbb{R}^n}|\xi|^2\left|(2\pi)^{(n/2)}a^{1-(1/2n)}\mathscr{F}_{Cl}[\mathbf{f}](\xi)\overline{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}]\left(\mathbf{r}S_sA_a\xi\overline{\mathbf{r}}\right)}\right|^2\mathrm{d}^n\xi\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}\right\}^{(1/2)}\\&\geq\frac{1}{2(2\pi)^{(n/2)}}\int_{\mathbb{R}^n\times\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\left|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right|^2\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}\mathrm{d}^n b.\end{aligned} \tag{71}$$

Equivalently, we have

$$\begin{aligned}&\left\{\int_{\mathscr{G}}|b|^2\left|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}\eta\right\}^{1/2}\times\left\{\int_{\mathbb{R}^n}|\xi|^2\left|\mathscr{F}_{Cl}[\mathbf{f}](\xi)\right|^2\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\left|\overline{\mathscr{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}]\left(\mathbf{r}S_sA_a\xi\overline{\mathbf{r}}\right)}\right|^2\mathrm{d}^n\xi\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}}\mathrm{d}^n\xi\right\}^{1/2}\\&\geq\frac{1}{2(2\pi)^n}\int_{\mathscr{G}}\left|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}\eta.\end{aligned} \tag{72}$$

Using the definition of C_Ψ in L. H. S and Corollary 1 in R. H. S, we obtain the desired result as follows

$$\left\|b\mathscr{C}\mathscr{S}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\right\|_{L^2\left(\mathscr{G},Cl_{(p,q)}\right)}\left\|\xi\mathscr{F}_{Cl}[\mathbf{f}](\xi)C_\Psi\right\|_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}\geq\frac{1}{2}\left|\langle\mathbf{f}(x)C_\Psi,\mathbf{f}(x)\rangle\right|_{L^2\left(\mathscr{G},Cl_{(p,q)}\right)}. \tag{73}$$

This completes the proof of Theorem 8. □

Remark 2. For real-valued C_Ψ, Theorem 5 boils down to

$$\left\|b\mathscr{C}\mathscr{S}_\Psi \mathbf{f}(a,\mathbf{r},s,b)\right\|_{L^2\left(\mathscr{G},Cl_{(p,q)}\right)}\left\|\xi\mathscr{F}_{Cl}[\mathbf{f}](\xi)\right\|_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}\geq\frac{\sqrt{C_\Psi}}{2}\left\|\mathbf{f}(x)\right\|^2_{L^2\left(\mathscr{G},Cl_{(p,q)}\right)}. \tag{74}$$

The classical Pitt's inequality expresses a fundamental relationship between a sufficiently smooth function $\mathbf{f}$ and the corresponding Clifford Fourier transform [19]. We derive the Pitt's type inequality for the proposed Clifford-valued shearlet transform (28). The Schwartz space on $C\ell_{(p,q)}$ algebras is given by

$$\mathbb{S}\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)=\left\{\mathbf{f}\in C^\infty\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right):\ \sup_{t\in\mathbb{R}^{(p,q)}}\left|t^\alpha\partial_t^\beta\mathbf{f}(t)\right|<\infty\right\}, \tag{75}$$

where $C^\infty(\mathbb{R}^{(p,q)},Cl_{(p,q)})$ is the class of smooth functions, and α,β denote multiindices, and ∂_t denotes the usual partial differential operator.

Theorem 9 (Pitt's inequality for $\mathscr{C}\mathscr{S}_\Psi$). *For any $\mathbf{f}\in\mathbb{S}(\mathbb{R}^{(p,q)},Cl_{(p,q)})$, the Pitt's inequality for the Clifford-valued shearlet transform (28) is given by*

$$\int_{\mathbb{R}^n}|\xi|^{-\lambda}\left|\mathscr{F}_{Cl}[\mathbf{f}](\xi)\right|^2\mathrm{d}^n\xi\leq\frac{C_\lambda}{(2\pi)^2}\int_{\mathscr{G}}|b|^\lambda\left|\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2C_\Psi^{-1}\mathrm{d}\eta, \tag{76}$$

where C_Ψ is the admissibility condition of Clifford-valued shearlet, and C_λ is given by

$$C_\lambda=\pi^\lambda\left[\frac{\Gamma'(n-\lambda/4)}{\Gamma(n+\lambda/4)}\right]^2,\quad 0\leq\lambda<n, \tag{77}$$

where $\Gamma(\cdot)$ denotes the well-known Euler's gamma function.

Proof. Considering $\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)$ as a function of the translation variable b, the Pitt's inequality in the Clifford Fourier domain implies 13:

$$\int_{\mathbb{R}^n}|\xi|^{-\lambda}\left|\mathscr{F}_{C\ell}\left[\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right](\xi)\right|^2\mathrm{d}^n\xi\leq\frac{C_\lambda}{(2\pi)^n}\int_{\mathbb{R}^n}|b|^\lambda\left|\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}^nb, \tag{78}$$

which upon integration with respect to the measure $(\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a/a^{n+1})$ yields

$$\begin{aligned}&\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\int_{\mathbb{R}^n}|\xi|^{-\lambda}\left|\mathscr{F}_{Cl}\left[\mathscr{C}\mathscr{S}_\psi\mathbf{f}(a,\mathbf{r},s,b)\right](\xi)\right|^2\mathrm{d}^n\xi\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}\\&\leq\frac{C_\lambda}{(2\pi)^n}\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\int_{\mathbb{R}^n}|b|^\lambda\left|\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}^nb\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}.\end{aligned} \tag{79}$$

Invoking Lemma 1, we can express the inequality (79) in the following manner:

$$\begin{aligned}&\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\int_{\mathbb{R}^n}|\xi|^{-\lambda}\left|(2\pi)^{(n/2)}a^{1-(1/2n)}\mathcal{F}_{Cl}[\mathbf{f}](\xi)\overline{\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})}\right|^2\mathrm{d}^n\xi\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}\\&\leq\frac{C_\lambda}{(2\pi)^n}\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\int_{\mathbb{R}^n}|b|^\lambda\left|\mathcal{C}\mathcal{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}^nb\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}.\end{aligned}\tag{80}$$

Equivalently, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}|\xi|^{-\lambda}\left|\mathcal{F}_{Cl}[\mathbf{f}](\xi)\right|^2\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\left|\mathcal{F}_{Cl}[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}](\mathbf{r}S_sA_a\xi\overline{\mathbf{r}})\right|^2\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}}\mathrm{d}^n\xi\\&\leq\frac{C_\lambda}{(2\pi)^{2n}}\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\int_{\mathbb{R}^n}|b|^\lambda\left|\mathcal{C}\mathcal{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}^nb\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}.\end{aligned}\tag{81}$$

Since Ψ is an admissible Clifford shearlet, inequality (81) boils down to

$$\int_{\mathbb{R}^n}|\xi|^{-\lambda}\left|\mathcal{F}_{Cl}[\mathbf{f}](\xi)\right|^2C_\Psi\mathrm{d}^n\xi\leq\frac{C_\lambda}{(2\pi)^{2n}}\int_{\mathcal{G}}|b|^\lambda\left|\mathcal{C}\mathcal{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}\eta.\tag{82}$$

which is the desired Pitt's inequality for the Clifford-valued shearlet transform. □

Remark 3. For $\lambda=0$, equality which holds in equation (76) is equivalent to equation (39).

Next, we shall formulate the logarithmic uncertainty principle for the Clifford-valued shearlet transform $\mathcal{C}\mathcal{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)$ given by equation (28).

Theorem 10 (Logarithmic uncertainty principle). *For any $\mathbf{f}\in\mathbb{S}(\mathbb{R}^{(p,q)},Cl_{(p,q)})$, the Clifford-valued shearlet transform $\mathcal{C}\mathcal{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)$ satisfies the following logarithmic estimate of the uncertainty inequality:*

$$\frac{1}{(2\pi)^n}\int_{\mathcal{G}}\left|\mathcal{C}\mathcal{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\ln|b|\mathrm{d}\eta+(2\pi)^n\int_{\mathbb{R}^n}\left|\mathcal{F}_{Cl}[\mathbf{f}](\xi)\right|^2C_\Psi\ln|\xi|\mathrm{d}^n\xi\geq\left(\frac{\Gamma'(n/4)}{\Gamma(n/4)}-\ln\pi\right)\left|\langle\mathbf{f}C_\Psi,\mathbf{f}\rangle\right|_{L^2(\mathcal{G},Cl_{(p,q)})},\tag{83}$$

provided the left hand side of this inequality is defined.

Proof. For the Clifford-valued function $\mathbf{f}\in\mathbb{S}(\mathbb{R}^{(p,q)},Cl_{(p,q)})$, the logarithmic uncertainty inequality in the Clifford Fourier domain yields [18]

$$\int_{\mathbb{R}^n}|\mathbf{f}(b)|^2\ln|b|\mathrm{d}^nb+(2\pi)^n\int_{\mathbb{R}^n}\left|\mathcal{F}_{Cl}[\mathbf{f}](\xi)\right|^2\ln|\xi|\mathrm{d}^n\xi\geq\left(\frac{\Gamma'(n/4)}{\Gamma(n/4)}-\ln\pi\right)\int_{\mathbb{R}^n}|\mathbf{f}(b)|^2\mathrm{d}^nb.\tag{84}$$

Upon replacing $\mathbf{f}(b)$ by $\mathcal{C}\mathcal{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)$ in the above inequality, we obtain

$$\int_{\mathbb{R}^n}\left|\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\ln|b|\mathrm{d}^n b+(2\pi)^n\int_{\mathbb{R}^n}\left|\mathscr{F}_{Cl}\left[\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right](\xi)\right|^2\ln|\xi|\mathrm{d}^n\xi \geq\left(\frac{\Gamma'(n/4)}{\Gamma(n/4)}-\ln\pi\right)\int_{\mathbb{R}^n}\left|\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}^n b. \tag{85}$$

Integrating equation (85) with respect to measure $(\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a/a^{n+1})$ and then invoking the Fubini theorem, we obtain

$$\int_{\mathscr{G}}\left|\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\ln|b|\mathrm{d}^n b\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}+(2\pi)^n\int_{\mathscr{G}}\left|\mathscr{F}_{Cl}\left[\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right](\xi)\right|^2\ln|\xi|\mathrm{d}^n\xi\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}} \geq\left(\frac{\Gamma'(n/4)}{\Gamma(n/4)}-\ln\pi\right)\int_{\mathscr{G}}\left|\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}^n b\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}. \tag{86}$$

Using Lemma 1, the inequality (86) can be further simplified as

$$\int_{\mathscr{G}}\left|\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\ln|b|\mathrm{d}^n b\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}+(2\pi)^n\int_{\mathscr{G}}\left|(2\pi)^{(n/2)}a^{1-(1/2n)}\mathscr{F}_{Cl}[\mathbf{f}](\xi)\overline{\mathscr{F}_{Cl}\left[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}\right]\left(\mathbf{r}S_sA_a\xi\overline{\mathbf{r}}\right)}\right|^2\ln|\xi|\mathrm{d}^n\xi\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}} \times\ln|\xi|\mathrm{d}^n\xi\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}\geq\left(\frac{\Gamma'(n/4)}{\Gamma(n/4)}-\ln\pi\right)\int_{\mathscr{G}}\left|\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}^n b\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{n+1}}. \tag{87}$$

Alternatively, the above inequality can be rewritten as

$$\int_{\mathscr{G}}\left|\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\ln|b|\mathrm{d}\eta+(2\pi)^{2n}\int_{\mathbb{R}^n}\left|\mathscr{F}_{Cl}[\mathbf{f}](\xi)\right|^2 \times\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathrm{Spin}(n)}\left|\mathscr{F}_{Cl}\left[\mathbf{r}\Psi(\cdot)\overline{\mathbf{r}}\right]\left(\mathbf{r}S_sA_a\xi\overline{\mathbf{r}}\right)\right|^2\frac{\mathrm{d}\mathbf{r}\mathrm{d}^{n-1}s\mathrm{d}a}{a^{(n^2-n+1/n)}}\ln|\xi|\mathrm{d}^n\xi \geq\left(\frac{\Gamma'(n/4)}{\Gamma(n/4)}-\ln\pi\right)\int_{\mathscr{G}}\left|\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\mathrm{d}\eta. \tag{88}$$

Noting that Ψ is admissible and using Corollary 1, we obtain the desired result as

$$\frac{1}{(2\pi)^n}\int_{\mathscr{G}}\left|\mathscr{C}\mathscr{S}_\Psi\mathbf{f}(a,\mathbf{r},s,b)\right|^2\ln|b|\mathrm{d}\eta+(2\pi)^n\int_{\mathbb{R}^n}\left|\mathscr{F}_{Cl}[\mathbf{f}](\xi)\right|^2C_\Psi\ln|\xi|\mathrm{d}^n\xi\geq\left(\frac{\Gamma'(n/4)}{\Gamma(n/4)}-\ln\pi\right)\left|\langle\mathbf{f}C_\Psi,\mathbf{f}\rangle\right|_{L^2\left(\mathscr{G},Cl_{(p,q)}\right)}. \tag{89}$$

This completes the proof of Theorem 10.

In the following, we establish a local-type uncertainty principle for the Clifford-valued sharelet transform $\mathscr{C}\mathscr{S}_\Psi\mathbf{f}$ defined by equation (28). More precisely, we shall demonstrate that the portion of $\mathscr{C}\mathscr{S}_\Psi$ lying outside some given set $\mathscr{M}$ of finite Lebesgue measure cannot be arbitrarily small. □

Theorem 11 (Concentration of $\mathscr{C}\mathscr{S}_\Psi$ in small sets). *Let $\Psi\in L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})$ be an admissible Clifford-valued shearlet satisfying $0<(|a|^{(1/2n)-1}\|\Psi\|^2\mu(\mathscr{M})/C_\Psi)<1$. Then, for any measurable subset $\mathscr{M}$ of $\mathscr{G}=\mathbb{R}^+\times\mathbb{R}^{n-1}\times\mathbb{R}^n\times Spin(n)$ and $\mathbf{f}\in L^2(\mathbb{R}^{(p,q)},Cl_{(p,q)})$, we have*

$$\left\|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right\|_{L^2\left(E^c,Cl_{(p,q)}\right)} \geq \sqrt{C_\Psi}\left(1-\frac{|a|^{(1/2n)-1}\mu(\mathscr{M})\|\Psi\|^2_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}}{C_\Psi}\right)^{1/2}\|f\|_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}, \tag{90}$$

where $\mu(\mathscr{M})$ denotes the measure of $\mathscr{M}$.

Proof. Using the definition of Clifford-valued shearlet transforms, we have

$$\begin{aligned}\left|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right|_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)} &= \left|a^{(1/2n)-1}\int_{\mathbb{R}^{(p,q)}}\mathbf{f}(x)\overline{\mathbf{r}}\overline{\Psi\left(A_a^{-1}S_s^{-1}\overline{\mathbf{r}}(x-b)\mathbf{r}\right)}\mathbf{r}\mathrm{d}^n x\right|_{L^2\mathbb{R}^{(p,q)},Cl_{(p,q)}}\\ &\leq |a|^{(1/2n)-1}\int_{\mathbb{R}^{(p,q)}}|\mathbf{f}(x)|\left|\overline{\mathbf{r}}\overline{\Psi\left(A_a^{-1}S_s^{-1}\overline{\mathbf{r}}(x-b)\mathbf{r}\right)}\mathbf{r}\right|\mathrm{d}^n x.\end{aligned} \tag{91}$$

By virtue of Holders inequality, we have

$$\left\|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right\|_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)} \leq |a|^{((1/2n)-1)}\|f\|_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}\|\Psi\|_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}. \tag{92}$$

On the other hand, we can write

$$\begin{aligned}\left\|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right\|^2_{L^2\left(\mathscr{G},Cl_{(p,q)}\right)\}} &= \int\iiint_{\mathscr{G}}\left|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right|^2_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}\mathrm{d}\eta\\ &= \int\iiint_{\mathscr{M}}\left|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right|^2_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}\mathrm{d}\eta + \int\iiint_{\mathscr{M}^c}\left|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right|^2_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}\mathrm{d}\eta\\ &\leq |a|^{(1/2n)-1}\mu(\mathscr{M})\|f\|^2_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}\|\Psi\|^2_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)} + \left\|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right\|^2_{L^2\left(E^c,Cl_{(p,q)}\right)}.\end{aligned} \tag{93}$$

Application of Corollary 1 for the real-valued C_Ψ implies that

$$C_\Psi\|f\|^2_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)} \leq |a|^{(1/2n)-1}\mu(\mathscr{M})\|f\|^2_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}\|\Psi\|^2_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)} + \left\|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right\|^2_{L^2\left(E^c,Cl_{(p,q)}\right)}, \tag{94}$$

or

$$\begin{aligned}\left\|\mathscr{C}\mathscr{S}_{\Psi}\mathbf{f}(a,\mathbf{r},s,b)\right\|_{L^2\left(E^c,Cl_{(p,q)}\right)} &\geq \left(C_\Psi - |a|^{(1/2n)-1}\mu(\mathscr{M})\|\Psi\|^2_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}\right)^{1/2}\|f\|_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}\\ &= \sqrt{C_\Psi}\left(1-\frac{|a|^{(1/2n)-1}\mu(\mathscr{M})\|\Psi\|^2_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}}{C_\Psi}\right)^{1/2}\|f\|_{L^2\left(\mathbb{R}^{(p,q)},Cl_{(p,q)}\right)}.\end{aligned} \tag{95}$$

This completes the proof of Theorem 11. □

6. Conclusion

In the present study, we formulated the notion of continuous Clifford-valued shearlet transform on the generalized geometric algebra $Cl_{p,q}$. The proposed transform has the advantage of efficiently handling Clifford-valued signals at various scales, positions and orientations while upholding the affine structure. Besides, studying the fundamental aspects of the Clifford-valued shearlet transform, the homogeneous approximation property is also investigated in detail. Nevertheless, some prominent uncertainty inequalities, such as the Hesienberg–Puali–Weyl logarithmic and local uncertainty principles are obtained at the end.

Data Availability

No data were generated.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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